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Aspects of $N = 4$ SYM

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Abstract: The properties of gauge-invariant composite operators and their correlation functions in $N = 4$ SYM are discussed in the analytic superspace formalism. A complete classification of the different types of operators in the theory is given. Operators can be either protected or unprotected according to whether they do not or do have anomalous dimensions, and the analytic superspace formalism allows one to identify which type a given operator is in a straightforward manner. A simple discussion is given of the behaviour of reducible multiplets at threshold. It is pointed out that there is a class of "semi-protected" operators which do not have anomalous dimensions but which do not necessarily have non-renormalised three-point functions when the other two operators in the correlator are protected, although two-point functions of such operators are non-renormalised. A complete discussion of superconformal invariants in analytic superspace is given. The paper includes a modified discussion of the transformation rules of analytic superfields which clarifies the $\mathrm{U}(1)_Y$ properties of operators and correlation functions and, in particular, explicit examples are given of three-point correlation functions which violate this symmetry. A tensor, $\mathcal{E}$, invariant under $\mathrm{SL}(n|m)$ but not under $\mathrm{GL}(n|m)$, is introduced and used in the discussion of $\mathrm{U}(1)_Y$ and in the construction of invariants.

Keywords: Extended Supersymmetry, Superspaces.

Contents

1. Introduction

Over the past few years there has been a substantial resurgence of interest in four-dimensional conformal field theories, particularly supersymmetric ones, largely inspired by the Maldacena conjecture relating IIB string theory on $AdS_5 \times S^5$ to $N = 4$ super Yang-Mills theory on Minkowski space, the conformal boundary of AdS_5 [1]. The subject now has an extensive literature and there are several review articles to which we refer the reader for lists of references [2, 3, 4].

In this paper we discuss the properties of gauge-invariant composite operators and their correlation functions in $N = 4$ superconformal field theory, that is to say $N = 4$ SYM, in four dimensions in the setting of analytic superspace. There have been many studies of such operators from various points of view but we believe that the analytic superspace formalism has some advantages. One is that holomorphic fields which transform under irreducible representations of the isotropy subgroup of the superconformal group which defines analytic superspace automatically transform irreducibly under the full superconformal group [5], and that all operators in the theory can be expressed as such holomorphic fields. A second is that it is easy to tell which operators will be protected in the quantum theory simply by looking at the representations they transform under and whether they can be written in terms of single trace 1/2 BPS operators (chiral primaries or CPOs) on analytic superspace [6]. We recall that, as well as short (BPS) operators, there are protected operators whose lack of anomalous dimensions was first deduced indirectly from the study of correlations functions [7]. In [6] it was noted that this phenomenon has a much simpler explanation in terms of representation theory in that the operators concerned are also subject to shortening conditions. The analytic superspace formalism makes it clear precisely which operators will remain short in the quantum theory. In the classical theory there can also be operators which transform under short representations but which turn out to be descendants of long operators and which therefore acquire the anomalous dimensions of the associated long operators in the quantum theory. This has been studied in detail for the case of 1/4 BPS operators [8, 9] and again can be understood simply in the analytic superspace formalism. This aspect of the theory can be looked at from the opposite point of view. One can consider the limit in which the anomalous dimension of a given long operator disappears. For some operators this limit results in a reducible representation. This reducibility at threshold was studied in great detail in [10] but it can also be understood very simply in analytic superspace.

The main technical development in this paper is an improved, and hopefully clearer, discussion of the transformation of operators in analytic superspace which places greater emphasis on the rôle of the so-called $\mathrm{U}(1)_Y$ "bonus symmetry" group [11]. There is some ambiguity concerning the representations of the isotropy group associated with long operators which is related to the violation of this symmetry for certain two- and three-point correlators involving such operators. We also draw attention to the existence of a class of "semi-protected" operators which are ambiguous in this sense and whose three-point correlators with other protected operators may not be non-renormalised. These operators, which can be either series A or series B, have the property that they saturate one unitarity bound but not both. Such representations were called "intermediate short" in [12]. A key rôle in the analysis of these operators and their correlators is played by a tensor, which we call $\mathcal{E}$, which is invariant under $\mathfrak{sl}(2|2)$ but not under $\mathfrak{gl}(2|2)$.[1] In addition, one can use this tensor to give a simple discussion of the $\mathrm{U}(1)_Y$ properties of superconformal invariants on analytic superspace.

The main properties of $N=4$ SYM that can be established using the analytic superspace formalism are: 1) the protection of operators, i.e. the lack of anomalous dimensions of operators transforming under short representations, 2) non-renormalisation theorems for two- and three-point functions of protected operators, 3) free-form and non-renormalisation of extremal and next-to-extremal correlators of protected operators and 4) the partial non-renormalisation of four-point functions of 1/2 BPS operators.[2] Of course, point 1 has been discussed from other points of view, see, for example, [13], but the analytic superspace formalism makes it clear precisely which operators are protected and allows a complete classification of all protected operators [6]. For point 2 we note that, although early derivations of results of this type were given either from the AdS point of view [14, 15] or in (mainly) perturbative field theory [16, 17], the analytic superspace formalism allows a compelling non-perturbative argument for the non-renormalisation of two- and three-point functions of protected operators on the field theory side [18]. This argument makes use of the reduction formula, introduced in [11], and can also be interpreted in terms of $\mathrm{U}(1)_Y$ symmetry as conjectured (and proved in the two-point case) in the same paper. It is applicable to general protected operators and not just the one-half BPS ones [19]. As we mentioned above, one has to take extra care when these correlators involve semi-protected operators. Points 3 and 4 will not be discussed in this paper (see [4] for references), although we again note that the analytic superspace formalism can be applied to extremal correlators involving general protected operators, not just one-half BPS superfields [19].

The $N=4$ superspace approach to $N=4$ SCFT is on-shell in the sense that the underlying $N=4$ SYM multiplet satisfies the field equations. In fact, one has to use these equations to establish the analyticity of operators such as the supercurrent. This circumstance has led some authors to criticise this approach and it is indeed the case that one cannot write down a path integral nor carry out perturbative calculations in $N=4$ superspace. However, the viewpoint we adopt is somewhat different. We are interested in analysing the constraints on the full non-perturbative theory due to superconformal invariance and we can view the $N=4$ formalism as a way of packaging the outcome of quantum calculations which can be carried out in any convenient manner, for example, using components. In the component formalism the superconformal algebra only closes on the fields modulo the equations of motion and gauge transformations and the standard gauge-fixing terms explicitly break supersymmetry. All of these technical difficulties

[1] Invariant tensors of this type exist for any $\mathfrak{sl}(n|m)$ algebra; they can be thought of as analogues of the ϵ-tensor in $\mathfrak{sl}(n)$.

[2] It should be noted that these results are predicated on the assumption that the superconformal Ward Identities hold in the quantum theory with composite operators; to our knowledge there is no rigorous proof of this currently in the literature.

can be overcome using the BRST/BV formalism for a combined BRST algebra which includes the BRST versions of both gauge and superconformal transformations, the latter involving space-time independent ghosts [20, 21]. In the study of correlation functions of gauge-invariant operators, however, one expects to recover the naive supersymmetry Ward identities. Moreover, the use of the equations of motion should be valid provided that one avoids coincident points. In principle, one might encounter difficulties due to contact terms when one uses the reduction formula which relates the derivative of an n-point function with respect to the coupling constant to an $(n+1)$-point function which has one integrated insertion of the on-shell action, but there is no evidence to date that any such difficulties arise. Moreover, it seems that the formalism correctly encodes the expected anomalies in the supercurrent due to the fact these are local so that the formally analytic expressions for the correlators have to be regularised in order to handle quantities that are ill-defined at coincident points.[3]

The organisation of the paper is as follows: in the next section we review the analytic superspace formalism and discuss the transformations of operators. The isotropy algebra includes two $\mathfrak{sl}(2|2)$ algebras which are analogous to the two $\mathfrak{sl}(2)$ spin algebras in ordinary Minkowski space. However, tensor fields transform under two $\mathfrak{gl}(2|2)$ algebras from which one cannot remove the supertrace in a canonical fashion. It is therefore natural to use $\mathfrak{gl}(2|2)$ representations and Young tableaux to describe tensor operators. These are not uniquely determined for long operators so that our presentation of the theory differs slightly from our earlier work which was largely concerned with protected operators. In section 3 we classify the operators of $N=4$ SYM with examples of each type of operator and discuss reducibility at threshold, and comment briefly on mixing for one-quarter BPS and other operators. In section 4 we discuss the $\mathrm{U}(1)_Y$ behaviour of two-and three-point functions. In particular, we exhibit explicit examples of correlators which are not invariant under this symmetry and which should therefore be subject to renormalisation effects. We show that the two-point functions of a semi-protected operator and its conjugate are non-renormalised even though there is a violation of $\mathrm{U}(1)_Y$ symmetry in the three-point function involving an additional supercurrent. We also compare our results with the earlier conjectures on $\mathrm{U}(1)_Y$ made in reference [22]. In section 5 we give a complete discussion of invariants in analytic superspace, both in coordinate language and in the Grassmannian formalism of [23]. The $\mathcal{E}$ tensor is used to give a simple explanation of the existence of superconformal invariants which are not $\mathrm{U}(1)_Y$ invariant. We give our conclusions in section 6. In an appendix we show how to explicitly convert a superconformal field on analytic superspace into a field on harmonic superspace.

2. Analytic superspace

Analytic superspaces were introduced in [24]; for a review see [25]. A harmonic superspace is a product of ordinary Minkowski space and a compact complex coset space of the internal

[3]For example, the three-point function of three supercurrent operators [17] is formally analytic but still encodes, amongst other anomalies, the usual triangle anomaly for the SU(4) currents due to the regularisation that is required when one has coincident points [14].

symmetry group. A field on harmonic superspace which is both Grasmmann analytic (generalised chiral) and analytic with respect to the internal manifold can be written as an unconstrained superfield on analytic superspace which has a reduced number of odd coordinates. This is similar to the way a chiral superfield can be written as an unconstrained superfield on chiral superspace. The general theory of such superspaces realised as coset spaces of complexified superconformal groups was developed in [26, 27] (see also [28]) and applied to four-dimensional super Yang-Mills theories in an earlier series of papers summarised in [29].

The analytic superspace we shall be using in this paper is a coset of the complexified $N = 4$ superconformal group PSL(4|4) with a parabolic isotropy group. We recall that a parabolic subgroup is one which contains the Borel subgroup. For the SL(n) Lie groups the latter can be thought of as the group of lower triangular $n \times n$ matrices, but in the supersymmetric case the Borel subgroup is no longer unique (up to conjugation). The group $(P)\,\mathrm{SL}(4|N)$ acts naturally from the left on $\mathbb{C}^{4|N}$, and the Borel subgroup can again be identified with the lower triangular matrices, but one obtains inequivalent Borel supergroups for different orderings of the basis elements of $\mathbb{C}^{4|N}$ with respect to Grassmann parity. The most convenient choice for applications to superconformal field theory has the form $(2|N|2)$ (i.e. (2 even|N odd|2 even)). For this choice, the parabolic subgroup which defines the analytic superspace we are interested in ($(4,2,2)$ analytic superspace) consists of supermatrices of the form:

$$\left(\begin{array}{cc|cccc|cc}
\bullet & \bullet & \bullet & \bullet & & & & \\
\bullet & \bullet & \bullet & \bullet & & & & \\
\hline
\bullet & \bullet & \bullet & \bullet & & & & \\
\bullet & \bullet & \bullet & \bullet & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\hline
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}\right) \tag{2.1}$$

where the bullets denote matrix elements which can be non-zero. The blank spaces can be thought of as representing local coordinates for analytic superspace, and these are therefore described by four sets of 2×2 matrices, two of which have even elements and two of which have odd elements. As can be seen from the above diagram, the coordinate supermatrix is not actually in standard form so that it is convenient to make another transposition of the basis (corresponding to the ordering $(2|2|2|2)$ of $\mathbb{C}^{4|4}$) in order to effect this. We then denote the coordinates by

$$X^{AA'} = \begin{pmatrix} x^{\alpha\alpha'} & \lambda^{\alpha a'} \\ \pi^{a\alpha'} & y^{aa'} \end{pmatrix}. \tag{2.2}$$

However, we shall adhere to the $(2|4|2)$ notation when labelling representations as this is more convenient for Minkowski superspace and also because this is the notation we have used in previous papers.

Geometrically, $(4,2,2)$ analytic superspace is the Grassmannian of planes of dimension $(2|2)$ in $\mathbb{C}^{4|4}$. Fields on this space are naturally what one might call generalised spinor fields, i.e they carry A and A' indices which are acted on in a linear way by the Levi subgroup of the isotropy group. This group consists of block diagonal elements of the type given in (2.1) (where the blocks have dimension $(2|2) \times (2|2)$). This is analogous to fields on Minkowski space carrying primed and unprimed spinor indices . In order to obtain representations of PSL(4|4) it turns out that the fields must carry the same number of primed and unprimed indices, while the representations will be unitary if all the indices are downstairs (covariant). The Levi subalegbra is $\mathfrak{ps}(\mathfrak{gl}(2|2) \oplus \mathfrak{gl}(2|2))$, but tensor fields will transform naturally under the two $\mathfrak{gl}(2|2)$s (acting on primed and unprimed indices). Since the unit $(n|n) \times (n|n)$ matrix has vanishing supertrace, it is not possible to remove the supertrace from these algebras in an invariant way, and so it is convenient to regard the tensor indices as $\mathfrak{gl}$ indices, despite the fact that the fields are labelled by representations of the two $\mathfrak{sl}$ subalgebras and an additional charge (the overall P being taken care of by having equal numbers of primed and unprimed indices as we have remarked above). In general there will be many $\mathfrak{gl}(2|2)$ representations which correspond to the same $\mathfrak{sl}(2|2)$ representation and this will play a rôle in the discussion of the $\mathrm{U}(1)_Y$ properties of various correlators to be discussed below. Before studying fields and correlators on analytic superspace we make a few remarks about ordinary four-dimensional (complex) Minkowski space which is again a Grassmannian; it is the space of 2-planes in $\mathbb{C}^4$.

2.1 Minkowski space

Complexified Minkowski space can be represented as a coset space of SL(4), where SL(4) is the complexified conformal group, with isotropy group consisting of matrices of the form

$$\begin{pmatrix} \bullet & \bullet & & \\ \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix}. \tag{2.3}$$

This space has coordinates $x^{\alpha\alpha'}$. The block-diagonal (Levi) subalgebra of the corresponding Lie algebra is $\mathfrak{s}(\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)) \sim \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathbb{C}$.[4] Conformal fields on Minkowski space have various $\mathfrak{sl}(2)$ indices α, α' and a dilation weight L which specify their transformation properties under the isotropy group. For irreducible representations of the isotropy algebra, only the block diagonal part, the Levi subalgebra, acts non-trivially. If we denote an element of $\delta g \in \mathfrak{sl}(4)$ by

$$\delta g = \begin{pmatrix} -A^{\alpha}{}_{\beta} & B^{\alpha\beta'} \\ -C_{\alpha'\beta} & D_{\alpha'}{}^{\beta'} \end{pmatrix} \tag{2.4}$$

then a conformal field $\mathcal{O}(x)$ will transform in the following way:

$$\delta\mathcal{O} = \mathcal{V}\mathcal{O} + \mathcal{R}(A(x))\mathcal{O} + \mathcal{R}'(D(x))\mathcal{O} + Q\Delta\mathcal{O} \tag{2.5}$$

[4] In real Minkowski space the two $\mathfrak{sl}(2)$ groups become complex conjugates of each other and $\mathbb{C}$ becomes $\mathbb{R}$.

where

$$
\begin{aligned}
(\mathcal{V}x)^{\alpha\alpha'} &= B^{\alpha\alpha'} + A^{\alpha}{}_{\beta}x^{\beta\alpha'} + x^{\alpha\beta'}D_{\beta'}{}^{\alpha'} + x^{\alpha\beta'}C_{\beta'\beta}x^{\beta\alpha'} \\
A(x)^{\alpha}{}_{\beta} &= A^{\alpha}{}_{\beta} + x^{\alpha\beta'}C_{\beta'\beta} \\
D(x)_{\alpha'}{}^{\beta'} &= D_{\alpha'}{}^{\beta'} + C_{\alpha'\alpha}x^{\alpha\beta'} \\
\Delta &= \mathrm{tr}(A + xC) = \mathrm{tr}(D + Cx)\,.
\end{aligned}
\tag{2.6}
$$

$\mathcal{R}$ and $\mathcal{R}'$ denote representations of the two $\mathfrak{gl}(2)$ algebras, and $Q = L - (J_1 + J_2)$ where J_1 and J_2 are the spin quantum numbers of the two $\mathfrak{sl}(2)$ algebras. It is assumed that the representations $\mathcal{R}$ and $\mathcal{R}'$ are given by Young tableaux with only one row, i.e they correspond to $\mathfrak{sl}(2)$ representations. Such a field (provided that it satisfies appropriate differential constraints) will correspond to an irreducible highest weight representation of $\mathfrak{sl}(4)$. A useful way of specifying such representations of simple (super) Lie groups is by assigning a number to each node of the corresponding (super) Dynkin diagram. These numbers give the highest weight of the representation which in turn uniquely specifies the entire irreducible representation. Furthermore one can specify a parabolic coset space by putting crosses through some nodes of the Dynkin diagram [30].[5] The resulting diagram can then be used to read off the transformation properties of the field on this space which carries the representation of the (super) conformal group in question.

In complex Minkowski space, therefore, representations of the complex conformal group $\mathfrak{sl}(4)$ can be encoded in the following diagram:

$$
\overset{n_1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{n_2}{\times}\!\!-\!\!\!-\!\!\!-\!\!\overset{n_3}{\bullet}
\tag{2.7}
$$

Here the three nodes are the Dynkin diagram for $\mathfrak{sl}(4)$, the numbers above the nodes specify the representation we are interested in, and the cross through the central node tells us we are interested in Minkowski space. If we wish to find out which field on Minkowski space carries this representation we can read this directly from the diagram. As we have already noted, to specify a conformal field on Minkowski space one needs to specify the representation of $\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)\oplus\mathbb{C}$ which acts linearly on the field. But the crossed through node splits the Dynkin diagram up into two $\mathfrak{sl}(2)$ Dynkin diagrams (consisting of single nodes) and a crossed through node. The numbers above each single node gives the representation of each $\mathfrak{sl}(2)$ and the number above the crossed node gives the $\mathbb{C}$ charge. Specifically, the above diagram corresponds to a field with $n_1 = 2J_1$ symmetrised unprimed spinor indices, $n_3 = 2J_2$ symmetrised primed spinor indices, and dilation weight $L = -n_2 - 1/2(n_1 + n_3)$. In addition, fields on Minkowski space must satisfy differential constraints in order to carry irreducible representations of the conformal group. For example the representation with Dynkin labels $n_1 = n_3 = 0,\ n_2 = -1$ corresponds to a massless scalar field and therefore satisfies the massless Klein-Gordon equation.

[5] For $\mathfrak{sl}(n)$ the crosses reduce the algebra to a direct sum of simple subalgebras and a number of abelian subalgebras, one for each cross; this diagram then represents the Levi subalgebra directly and the parabolic is obtained by filling out block diagonal elements of $\mathfrak{sl}(n)$ to block lower-triangular matrices

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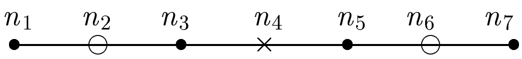
A given $\mathfrak{sl}(2)$ tableau with m boxes can be described in $\mathfrak{gl}(2)$ by any two-row tableau of the form $\langle m_1 m_2\rangle$ where m_1 and m_2 specify the number of boxes in the second and first rows of the $\mathfrak{gl}(2)$ Young tableau respectively (this unusual notation ties in with the supersymmetric case later), and where $m_2 - m_1 = m$. This means that, instead of using the representations $\mathcal{R}$ and $\mathcal{R}'$ corresponding to single-row tableaux with n_1 and n_3 boxes respectively, we could equivalently use a field specified by the labels $(\langle m_1 m_2\rangle\langle m_1' m_2'\rangle\tilde{Q})$, where we use two-row Young tableaux to specify the representations of the two $\mathfrak{gl}(2)$ algebras. In order to describe the same representation as before we must have $m_1 - m_2 = n_1$, $m_1' - m_2' = n_3$ while $\tilde{Q} = Q + (m_1 + m_1')$. In other words, we may use a field which transforms under any representations $\tilde{\mathcal{R}}, \tilde{\mathcal{R}}'$ which correspond to the same representations as $\mathcal{R}, \mathcal{R}'$ under the $\mathfrak{sl}(2)$ subalgebras of the $\mathfrak{gl}(2)$s, provide that we change the value of the charge Q appropriately.

This is a rather trivial manoeuvre in the present example, but it will be useful in the $N = 4$ case as we shall see below. The main reason for the difference is that $N = 4$ analytic superspace also carries an action of $\mathrm{U}(1)_Y$, and the charge of a given field under this group will be different for different choices of $\mathfrak{gl}(2|2)$ tensor representations.

2.2 Fields on analytic superspace

Representations of $\mathrm{SL}(4|N)$ are specified by the quantum numbers $(L, R, J_1, J_2, a_1, \ldots, a_N)$ where L is the dilation weight, R is the R-charge, J_1 and J_2 are spin labels and $(a_1, \ldots, a_N)$ are $\mathrm{SL}(N)$ Dynkin labels. In $N = 4$, representations of $\mathrm{PSL}(4|4)$ are representations of $\mathrm{SL}(4|4)$ which satisfy the constraint $R = 0$. They are therefore specified by the quantum numbers $(L, J_1, J_2, a_1, a_2, a_3)$. The Dynkin diagram for superfields on analytic superspace carrying such representations of the $N = 4$ superconformal group is



(2.8)

We see that the single cross splits the Dynkin diagram into two smaller Dynkin diagrams which each represent $\mathfrak{sl}(2|2)$, the crossed-through node again representing a $\mathbb{C}$ charge. The white nodes correspond to odd roots and the number and position of such nodes depends on the choice of basis of $\mathbb{C}^{4|4}$ that has been chosen; the one we are using here corresponds to the ordering $(2|N|2)$. The two $\mathfrak{sl}(2|2)$ sub-Dynkin diagrams have a single central white node and correspond to the usual basis ordering $(2|2)$ of $\mathbb{C}^{(2|2)}$.

The quantum numbers are related to the super Dynkin labels by

$$
\begin{aligned}
n_1 &= 2J_1 \\
n_2 &= \frac{1}{2}(L - R) + J_1 + \frac{M}{4} - M_1 \\
n_{2+i} &= a_i \qquad (i = 1 \ldots 3) \\
n_6 &= \frac{1}{2}(L + R) + J_2 - \frac{M}{4} \\
n_7 &= 2J_2
\end{aligned}
\tag{2.9}
$$

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where M is the total number of boxes in the Young tableau of the internal $\mathfrak{sl}(4)$ representation, and M_1 is the number of boxes in the first row of this tableau, i.e.

$$M = \sum_{k=1}^{3} k\, a_k \qquad M_1 = \sum_{k=1}^{3} a_i\,. \tag{2.10}$$

From (2.9) we see that

$$R = \frac{1}{2}(n_1 - 2n_2 - n_3) + \frac{1}{2}(n_5 + 2n_6 - n_7) \tag{2.11}$$

so that the representations that we are interested in, which have $R = 0$, satisfy the constraint

$$I := n_3 + 2n_2 - n_1 = n_5 + 2n_6 - n_7\,. \tag{2.12}$$

We shall see in a moment that this corresponds to tensors on analytic superspace which have the same total number, I, of primed and unprimed indices. In terms of the Dynkin labels the dilation weight L is given by

$$L = \sum_{i=2}^{6} n_i - (n_1 + n_7)\,. \tag{2.13}$$

In the field theory context we are interested in unitary representations (of the real superconformal group) [31]; there are three series of such operators (for general N) and they satisfy certain unitarity bounds:

$$\begin{array}{llllll}
\text{Series A:} & L \geq 2 + 2J_1 + 2M_1 - \dfrac{m}{2} & L \geq 2 + 2J_2 + \dfrac{M}{2} & & \\
\text{Series B:} & L = \dfrac{M}{2}\,; & L \geq 1 + M_1 + J_1\,, & J_2 = 0 & \text{or} \\
 & L = 2m_1 - \dfrac{M}{2}\,; & L \geq 1 + M_1 + J_2\,, & J_1 = 0 & \\
\text{Series C:} & L = M_1 = \dfrac{M}{2} & J_1 = J_2 = 0\,. & &
\end{array} \tag{2.14}$$

For $N = 4$, these bounds can be rewritten in terms of the Dynkin labels as follows:

$$\begin{array}{llll}
\text{Series A:} & n_2 \geq n_1 + 1\,, & n_6 \geq n_7 + 1 & \\
\text{Series B:} & n_2 \geq n_1 + 1\,, & n_6 = 0\,, & n_7 = 0 \\
\text{Series C:} & n_2 = 0\,, & n_6 = 0\,, & n_1 = n_7 = 0
\end{array} \tag{2.15}$$

(or $n_1 \to n_7$, $n_2 \to n_6$ for series B.)

These bounds will be satisfied if the tensor representations of the two $\mathfrak{gl}(2|2)$ algebras carried by the superfields correspond to proper Young tableaux (see below) for covariant tensors.

As in the case of Minkowski space, if we are given a representation of the superconformal group, we can find a corresponding analytic superfield which carries that representation: the Dynkin labels specifying the representation of $\mathfrak{sl}(4|4)$ in question also specify the representation of $\mathfrak{sl}(2|2) \oplus \mathfrak{sl}(2|2) \oplus \mathbb{C}$ which the superfields carry linearly (in practice as tensors with superindices). To carry this out we need to know how to convert from $\mathfrak{sl}(2|2)$ Dynkin labels to Young tableaux so that we can explicitly write down the tensor fields. However, as we remarked above, it is easier to consider the tensor indices A, A' as $\mathfrak{gl}(2|2)$ indices, and so we shall use Young tableaux for this bigger group.

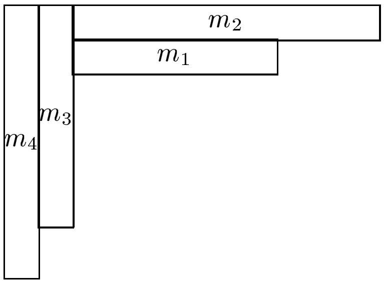
An important difference between fields on Minkowski space and fields on analytic superspace is that on analytic superspace all holomorphic, irreducible tensor fields automatically carry irreducible representations; no differential constraints need to be imposed as in the case of fields on Minkowski space [5]. This is very similar to what happens in twistor theory where irreducible representations of the conformal group (for example massless fields in Minkowski space) are given by cohomology classes on twistor space without constraints (see [30]). In the supersymmetric case, however, one only needs the zeroth cohomology classes (i.e. holomorphic superfields) on analytic superspaces. This property enables one to write down solutions to the superconformal Ward identities without having to solve differential equations.

2.3 $\mathfrak{gl}(2|2)$ Young tableaux

Superconformal operators on $(4,2,2)$ analytic superspace have superindices which can carry either $\mathfrak{sl}(2|2)$ or $\mathfrak{gl}(2|2)$ representations (as compared to conformal operators on Minkowski space which have spinor indices carrying representations of $\mathfrak{sl}(2)$ or $\mathfrak{gl}(2)$). Representations of the left $\mathfrak{sl}(2|2)$ algebra are determined by the three Dynkin labels (n_1, n_2, n_3) and those of the right $\mathfrak{sl}(2|2)$ algebra by the three labels (n_7, n_6, n_5) corresponding to the left and right halves respectively of the diagram (2.8).

As in the case of purely bosonic simple Lie algebras, all finite dimensional representations of $\mathfrak{sl}(2|2)$ with integer coefficients can be obtained by tensoring together copies of the fundamental and/or anti-fundamental representation (which are inequivalent for supergroups) and taking certain (anti-)symmetric combinations which are given in a Young tableaux [32]. The Young tableaux is interpreted just as for purely bosonic groups (the number of boxes corresponds to the number of indices, rows correspond to symmetrised indices and columns to anti-symmetrised) with the difference that the terms symmetrised and anti-symmetrised are generalised to take into account of the fact that some of the indices are odd. This implies that there is no limit to the number of boxes in a given column. However, the tableau in the form of a 3×3 square is identically zero, for symmetry reasons, and so the tableaux are restricted to have the form given below, with only the two left-most columns having arbitrary length. We note that the tensor product of representations can be computed by multiplying tableaux together. The rules for this are the same as in the bosonic case, but any tableau in the product which contains a 3×3 square sub-tableau can be discarded.

The unitarity conditions (2.14) imply that the unitary representations are those obtained by tensoring the anti-fundamental representation of $\mathfrak{sl}(2|2)$. However, as we remarked above, it is convenient to work with $\mathfrak{gl}(2|2)$ tableaux. The most general such Young tableau that can be obtained has the form:



(2.16)

where m_i denotes the number of boxes in the indicated sections of the Young tableaux. We will denote this Young tableau by $\langle m_1, m_2, m_3, m_4 \rangle$. A diagram will be said to be proper if $m_1 \leq m_2$, $m_3 \leq m_4$ and if $m_2 \neq 0$, $m_3 \geq 1$, while if $m_1, m_2 \neq 0$, $m_3 \geq 2$. This diagram is related to the Dynkin labels $\{n_i\}$ by

$$\begin{aligned} m_4 - m_3 &= n_3 \\ m_3 + m_2 &= n_2 \\ m_2 - m_1 &= n_1 \,. \end{aligned} \tag{2.17}$$

We get a similar Young tableau $\langle m'_1, m'_2, m'_3, m'_4 \rangle$ for the right $\mathfrak{sl}(2|2)$ algebra with

$$\begin{aligned} m'_4 - m'_3 &= n_5 \\ m'_3 + m'_2 &= n_6 \\ m'_2 - m'_1 &= n_7 \,. \end{aligned} \tag{2.18}$$

Since there are only three numbers which determine each representation of $\mathfrak{sl}(2|2)$ and there are four numbers determining the Young tableau, in general we can have different tableaux giving the same $\mathfrak{sl}(2|2)$ representation. This must be the case as the Young tableaux also give representations of $\mathfrak{gl}(2|2)$ and different $\mathfrak{gl}(2|2)$ representations may correspond to the same $\mathfrak{sl}(2|2)$ representation (just as in the case of Minkowski space we can have different $\mathfrak{gl}(2)$ representations corresponding to the same $\mathfrak{sl}(2)$ representation). Specifically,

$$\langle m_1, m_2, m_3, m_4 \rangle \sim \langle m_1 - m, m_2 - m, m_3 + m, m_4 + m \rangle \tag{2.19}$$

for any integer m, such that $2 - m_3 \leq m \leq m_1$ (here '$\sim$' means 'corresponds to the same $\mathfrak{sl}(2|2)$ representation as'.) In particular for any $\mathfrak{sl}(2|2)$ representation there is always a Young tableau with $m_1 = 0$ corresponding to this representation. We refer to this choice of Young tableau as the canonical form of the representation.

One way of interpreting this is in terms of a tensor which is invariant under $\mathfrak{sl}(2|2)$ but not under $\mathfrak{gl}(2|2)$. This can be considered to be an analogue of the ϵ-tensor. In $\mathfrak{sl}(2)$ the tableau $\langle 1, 1 \rangle$ (two boxes in a column) is equivalent to the trivial tableau, so that one can deduce the existence of a two-index antisymmetric tensor which is invariant under $\mathfrak{sl}(2)$, but not under $\mathfrak{gl}(2)$. In $\mathfrak{sl}(2|2)$ the tableau $\langle 1, 1, 2, 2 \rangle$ (two rows each with three boxes) is equivalent to the tableau $\langle 0,0,3,3\rangle$ (two columns each with three boxes), so that there is a tensor with 6 covariant indices in the symmetry pattern of the first tableau and 6 contravariant indices in the symmetry pattern of the second tableau which is invariant under $\mathfrak{sl}(2|2)$ but not under $\mathfrak{gl}(2|2)$; it transforms by a factor of the superdeterminant under the bigger group. We shall refer to this tensor as $\mathcal{E}$. There is also an inverse tensor for which the covariant and contravariant symmetry patterns are interchanged. If we start from a tensor in a representation with $m_1 = 0$ we can obtain a tensor in the representation with $m_1 = m$ by applying $\mathcal{E}$ m times (together with appropriate projections). We shall write this operation schematically as $\mathcal{E}(m)$.

As well as the tensor representations of $\mathfrak{sl}(2|2)$, with integer Dynkin labels $[a,(a+b),c]$, there are also representations with b non-integral. We call these quasi-tensor representations. In the superconformal field theory context unitarity implies b must be real and greater than one, with the limit $b = 1$ (discussed in subsection 3.2) giving rise to reducible representations. For simplicity we can think about the case $[0,b,0]$. For b a positive integer the canonical Young tableau for this representation is $\langle 0,0,b,b\rangle$, i.e. a two-column tableau with b boxes in each column. The other possible tableaux will be of the form $\langle k,k,b-k,b-k\rangle$. The representation with $b = 1$ is short but all of the others, with $b \geq 2$, have the same dimension. We can therefore realise such representations on fields which may be chosen to have the same index pattern as $\langle 0,0,2,2\rangle$. If $p = b+2$, we shall denote such a field by $\mathcal{O}[p,0]$. Instead of using the canonical tableau we could have used, for example, the tableau $\langle 1,1,b-1,b-1\rangle$. Again all such tensors have the same number of components and can therefore be represented on fields which have the same index pattern as the tableau $\langle 0,0,2,2\rangle$. We denote such a field by $\tilde{\mathcal{O}}[p-1,1]$. The two representation spaces are isomorphic and are related by an isomorphism which we can again denote by $\mathcal{E}$, so $\tilde{\mathcal{O}}[p-1,1] = \mathcal{E}\mathcal{O}[p,0]$. All of this continues to make sense for b real, $b > 1$, so that we can extend the notion of the $\mathcal{E}$ tensor to the case of quasi-tensors.

2.4 Field transformations in analytic superspace

For any unitary irreducible representation we are now in a position to be able to give a superfield on analytic superspace which carries this representation. If the representation has Dynkin labels $[n_1 n_2 \ldots n_7]$ (which can be obtained from the usual quantum numbers from (2.9)) then we denote by $\mathcal{R}, \mathcal{R}'$ the left and right representation of $\mathfrak{sl}(2|2)$ respectively. In practice these will be specified by tensor indices, symmetrised according to Young tableaux as described above. There is some ambiguity as to how to do this, so we will choose the canonical Young tableau, with $m_1 = 0$, that is

$$\mathcal{R} = \langle 0, n_1, n_2 - n_1, n_3 + n_2 - n_1\rangle \tag{2.20}$$

$$\mathcal{R}' = \langle 0, n_7, n_6 - n_7, n_5 + n_6 - n_7\rangle \,. \tag{2.21}$$

The total number of indices of the representation $\mathcal{R}$, given by the number of boxes in the corresponding Young tableau, is

$$I := n_3 + 2n_2 - n_1 = n_5 + 2n_6 - n_7 \,. \tag{2.22}$$

This equality occurs because we are only considering representations of SL(4|4) with zero R-charge (equivalent to studying representations of PSL(4|4)) and this translates into the condition that the number of indices of the $\mathcal{R}$ and $\mathcal{R}'$ representations is the same.

We denote a general operator by $\mathcal{O}^Q_{\mathcal{R}\mathcal{R}'}$ where $Q = L - J_1 - J_2$ for canonical tableaux. Under an infinitesimal superconformal transformation specified by an infinitesimal $\mathfrak{sl}(4|4)$ matrix

$$\delta g = \begin{pmatrix} -A^A{}_B & B^{AB'} \\ -C_{A'B} & D_{A'}{}^{B'} \end{pmatrix}, \tag{2.23}$$

where each entry is a $\mathfrak{gl}(2|2)$ matrix and where $\mathrm{str}(A) = \mathrm{str}(D)$, an operator $\mathcal{O}^Q_{\mathcal{R}\mathcal{R}'}$ transforms as

$$\delta\mathcal{O}^Q_{\mathcal{R}\mathcal{R}'} = \mathcal{V}\mathcal{O}^Q_{\mathcal{R}\mathcal{R}'} + \mathcal{R}(A(X))\mathcal{O}^Q_{\mathcal{R}\mathcal{R}'} + \mathcal{R}'(D(X))\mathcal{O}^Q_{\mathcal{R}\mathcal{R}'} + Q\Delta\mathcal{O}^Q_{\mathcal{R}\mathcal{R}'}\,. \tag{2.24}$$

where

$$\mathcal{V}X = B + AX + XD + XCX \tag{2.25}$$

$$A(X) = A + XC \tag{2.26}$$

$$D(X) = D + CX \tag{2.27}$$

$$\Delta = \mathrm{str}(A + XC) = \mathrm{str}(D + CX)\,. \tag{2.28}$$

Here $\mathcal{V}$ is the vector field generating the transformation. $A(X)$ and $D(X)$ are $\mathfrak{gl}(2|2)$ matrices rather than $\mathfrak{sl}(2|2)$ matrices, but the Young tableaux define representations of this group as well and so $\mathcal{R}(A(X))$, $\mathcal{R}(D(X))$ make sense. Note that the unit $(4|4)\times(4|4)$ matrix does not act on analytic superspace, and if we only consider superfields that have equal numbers of left and right superindices, then the identity matrix does not act on the superfield indices either so we automatically obtain representations of $\mathfrak{psl}(4|4)$.

2.5 $\mathrm{U}(1)_Y$ and PGL(4|4) transformations

Although the unit matrix does not act on analytic superspace there is nevertheless a non-trivial action of the algebra $\mathfrak{pgl}(4|4)$ which extends $\mathfrak{psl}(4|4)$ by an abelian algebra. We shall refer to the corresponding abelian group as $\mathrm{U}(1)_Y$ even though in the complexified setting it is really a $\mathbb{C}^* := \mathbb{C}\setminus\{0\}$ group. In the free theory, this group is a symmetry, and we can extend the $\mathfrak{psl}(4|4)$ transformations to $\mathfrak{pgl}(4|4)$ transformations. For the infinitesimal transformations we are considering this simply amounts to removing the condition $\mathrm{str}(\delta g) = 0$ (that is, we drop the constraint $\mathrm{str}(A) = \mathrm{str}(D)$). The transformation of X looks exactly as before (2.25), but now the matrices A and D are unrestricted.

If we consider an $\mathfrak{gl}(4|4)$ transformation given by a diagonal matrix with

$$A \sim \frac{1}{2}\begin{pmatrix} a_o I_{(2|2)} & 0 \\ 0 & a_1 I_{(2|2)} \end{pmatrix} \tag{2.29}$$

where $I_{(2|2)}$ denotes the unit tensor acting on $\mathbb{C}^{2|2}$, and similarly for D, we find the transformations

$$\delta x = sx \tag{2.30}$$

$$\delta\lambda = \frac{1}{2}(s + s' + \Delta')\lambda \tag{2.31}$$

$$\delta\pi = \frac{1}{2}(s + s' - \Delta')\lambda \tag{2.32}$$

$$\delta y = s' y \tag{2.33}$$

where

$$s = \frac{1}{2}(a_o + d_o) \tag{2.34}$$

$$s' = \frac{1}{2}(a_1 + d_1) \tag{2.35}$$

$$\Delta' = \frac{1}{2}(a_o - a_1 - (d_o - d_1)) = -\frac{1}{2}\mathrm{str}(\delta g)\,. \tag{2.36}$$

The parameters s and s' correspond to dilations and internal dilations respectively, and we can identify Δ' as the $\mathrm{U}_Y(1)$ parameter which acts only on the odd variables. Note that

$$\Delta' = \frac{1}{2}(\mathrm{str}(A + XC) - \mathrm{str}(D + CX)) = \frac{1}{2}(\mathrm{str}(A - D))\,. \tag{2.37}$$

There is some ambiguity in the extension of the definition of Δ (equation (2.28)) to this case. We shall take it to be

$$\Delta = \frac{1}{2}(\mathrm{str}(A + XC) + \mathrm{str}(D + CX))\,. \tag{2.38}$$

This definition is different to the one used in our previous papers, but has the advantage that the free field-strength superfield can be assigned zero $\mathrm{U}(1)_Y$ charge, Q'. We recall that this superfield, which has quantum numbers $(L, J_1, J_2, a_1, a_2, a_3) = (1,0,0,0,1,0)$, or, equivalently, super Dynkin labels [0001000], is represented on analytic superspace as a single-component superfield W with $Q = 1$. In other words, W transforms as

$$\delta W = \mathcal{V}W + \Delta W \tag{2.39}$$

under a $\mathfrak{pgl}(4|4)$ transformation. Here $\mathcal{V}$ is the vector field which generates $\mathfrak{pgl}(4|4)$ transformations on analytic superspace: it has the same form as (2.25) but with the constraint $\mathrm{str}A = \mathrm{str}D$ dropped.

Let us now consider the transformation of an arbitrary operator $\mathcal{O}^{QQ'}_{\mathcal{R}\mathcal{R}'}(X)$ on analytic superspace under the extended algebra $\mathfrak{pgl}(4|4)$. The field is specified by the ten labels $(\langle m_1 m_2 m_3 m_4\rangle,\ \langle m'_1 m'_2 m'_3 m'_4\rangle, Q, Q')$, but one only needs seven to specify a representation of this group. Since the number of primed and unprimed indices are equal we have $\sum m_i = \sum m'_i$, so that only 9 of the labels are independent. Furthermore, we can alter the representations $\mathcal{R}$ and $\mathcal{R}'$ without changing the way the field transforms under the two $\mathfrak{sl}(2|2)$ algebras, and still have the same representation provided that the charges are also adjusted. Thus two more of the labels are redundant and the representation is therefore specified by seven quantum numbers as expected. The transformation of the field is given by

$$\delta\mathcal{O}^{QQ'}_{\mathcal{R}\mathcal{R}'} = \left(\mathcal{V} + \mathcal{R}(A(X)) + \mathcal{R}'(D(X)) + Q\Delta + Q'\Delta'\right)\mathcal{O}^{QQ'}_{\mathcal{R}\mathcal{R}'}\,. \tag{2.40}$$

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To see what happens when we change the $\mathfrak{gl}(2|2)$ representations, suppose that $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are two representations given by the Young tableaux $\langle m_1 m_2 m_3 m_4 \rangle$ and $\langle \tilde{m}_1 \tilde{m}_2 \tilde{m}_3 \tilde{m}_4 \rangle$ respectively where $\tilde{m}_1 = m_1 + m, \tilde{m}_2 = m_2 + m, \tilde{m}_3 = m_3 - m, \tilde{m}_4 = m_4 - m$, so that the two representations correspond to the same representation in $\mathfrak{sl}(2|2)$. If $A \in \mathfrak{gl}(2|2)$ one has

$$\tilde{\mathcal{R}}(A) = \mathcal{E}(m)\mathcal{R}(A)\mathcal{E}(m)^{-1} + m \operatorname{str} A \tag{2.41}$$

and so we find (with a similar change of representation for the primed algebra),

$$\tilde{\mathcal{R}}(A(X)) + \tilde{\mathcal{R}}'(D(X)) + \tilde{Q}\Delta + \tilde{Q}'\Delta' \sim \mathcal{R}(A(X)) + \mathcal{R}'(D(X)) + Q\Delta + Q'\Delta' \tag{2.42}$$

where

$$\tilde{Q} = Q - (m + m') ; \qquad \tilde{Q}' = Q' - (m - m') . \tag{2.43}$$

So we can change the $\mathfrak{gl}(2|2)$ representations that a field transforms according to, while preserving the $\mathfrak{sl}(2|2)$ representations, provided that we adjust the charges accordingly.

In the free $N = 4$ SYM theory, any unitary representation can be written on analytic superspace in terms of free Maxwell superfields such as W and derivatives $\partial_{A'A}$ and will have the schematic form $\partial^I W^Q$. We can form irreducible operators by projecting the I primed and unprimed indices onto irreducible $\mathfrak{gl}(2|2)$ representations $\mathcal{S}, \mathcal{S}'$, thereby obtaining an operator which we denote by $\mathcal{O}^Q_{\mathcal{S}\mathcal{S}'}$. It will transform under $\mathfrak{pgl}(4|4)$ transformations according to the formula (2.40) with $\mathcal{R}, \mathcal{R}'$ replaced by $\mathcal{S}, \mathcal{S}'$ and $Q' = 0$. A simple example of such an operator is the supercurrent $T := \operatorname{tr}(W^2)$; this has no indices and hence corresponds to trivial tableaux, and since it has two powers of W it has $Q = 2$. The Konishi superfield, on the other hand, is given in the free theory by

$$\mathcal{O}_{AB,A'B'} = \partial_{(A(A'} W \partial_{B)B')} W - \frac{1}{6}\partial_{(A(A'}\partial_{B)B')}(W^2) \tag{2.44}$$

where both pairs of superindices are symmetrised. So there are $Q = 2$ Ws and it has left and right Young tableaux $\langle 0, 0, 1, 1 \rangle$.

For any such operator we can change the tableaux $\mathcal{S}, \mathcal{S}'$ determined by the symmetry properties of the indices to the corresponding canonical tableaux $\mathcal{R}, \mathcal{R}'$ for which $m_1 = m'_1 = 0$; we can thereby obtain an equivalent operator which will describe the same representation of $\mathfrak{pgl}(4|4)$ provided that we change the charges Q and Q'. So we have

$$\mathcal{O}^Q_{\mathcal{S}\mathcal{S}'} \sim \mathcal{O}^{\tilde{Q}Q'}_{\mathcal{R}\mathcal{R}'} \tag{2.45}$$

where, if $\mathcal{S}$ corresponds to the tableaux $\langle m_1 \dots m_4 \rangle$, $\mathcal{R}$ corresponds to the tableau $\langle 0, m_2 - m_1, m_3 + m_1, m_4 + m_1 \rangle$, and similarly for $(\mathcal{S}', \mathcal{R}')$, and where

$$\tilde{Q} = Q + (m_1 + m'_1) ; \qquad Q' = m_1 - m'_1 . \tag{2.46}$$

Thus any tensor operator of this type has well-defined properties under the group PGL$(4|4)$. We shall say that the operator $\mathcal{O}^Q_{\mathcal{S}\mathcal{S}'}$ has $\mathrm{U}(1)_Y$ charge $m_1 - m'_1$. In fact, the $\mathrm{U}(1)_Y$ change of the highest weight state is $m_1 - m'_1 + (J_1 - J_2)$.

This can be generalised to the interacting quantum theory. The quantum numbers of a given operator will not change, with the possible exception of the dimension which may become anomalous. So the Young tableaux and Q charge can be assigned in a straightforward way, and again we will obtain operators with $Q' = m_1 - m_1'$. However, not all operators in the free theory can be straightforwardly generalised to the interacting classical theory as fields on analytic superspace. This is because the field strength superfield W transforms under the adjoint representation of the gauge group and is covariantly analytic. It is therefore no longer a holomorphic field on analytic superspace. Moreover, there is no notion of a gauge covariant derivative $\nabla_{A'A}$ on this space. The operators that cannot be written in terms of gauge-invariant products of Ws and derivatives of these in the interacting theory can be either long operators like the Konishi operator or descendants of long operators. Note that, for example, the interacting Konishi operator can be written as a field, component by component, on analytic superspace even though it cannot be written in terms of derivatives and Ws.

The protected operators are those which can be written in terms of derivatives and gauge-invariant products of Ws and which are in shortened representations (possibly series A). Such operators saturate a unitarity bound and have either $n_2 = n_1 + 1$ or both $n_1 = n_2 = 0$ and/or similar constraints for n_7, n_6. When constraints of this type are satisfied for either $\mathfrak{gl}(2|2)$ tableau then it is easy to verify that this tableau must be in canonical form, i.e. $m_1 = 0$. For more general operators this is not the case, but we can nevertheless bring the tableaux to canonical form as long as we change the charges Q and Q'. If the original, naturally defined, operator has $m_1 \neq m_1'$, then the form of the operator with canonical tableaux will have non-zero $\mathrm{U}(1)_Y$ charge given by $m_1 - m_1'$. However, as we shall see below, there is nothing in principle to stop long operators which transform under the same representation of PSL(4|4), but which have different $\mathrm{U}(1)_Y$ charges, mixing in the quantum theory, so that $\mathrm{U}(1)_Y$ charge will only be a good quantum number for the protected operators

3. Classification of operators

In this section we summarise the results that can be established for various operators in $N = 4$ SYM using the analytic superspace formalism. It is possible to write down explicitly all operators in the free theory on analytic superspace as unconstrained superfields and this perhaps provides the simplest way of finding the full spectrum of gauge invariant operators in the theory. It would be very interesting to compare this with recent results concerning the spectrum of string theory in $AdS_5 \times S^5$ [33].

3.1 Classification

We begin with the free theory. In this case there are no anomalous dimensions and one can explicitly construct examples of operators which transform according to any irreducible representation of the superconformal group from products of free field strength tensors and analytic superspace derivatives [34]. If we now suppose we have $N_c^2 - 1$ of these which transform under the adjoint representation of a rigid $\mathrm{SU}(N_c)$ symmetry group and we demand that our operators be invariant under this group, then not all representations of the superconformal group will be permissible.

Now let us consider the classical interacting theory where the $\mathrm{SU}(N_c)$ group is taken to be a gauge group and the operators are required to be gauge-invariant. Among the operators listed above there will be those which involve derivatives acting directly on Ws. However, since there is no gauge-covariant derivative on analytic superspace, such operators will not generalise to the interacting case as operators on analytic superspace. These operators can be of two types: operators which become long, such as the Konishi operator, and descendants of long operators which will not exist as separate operators in the interacting quantum theory. The remaining operators can be constructed from products of the single trace 1/2 BPS operators and ordinary analytic superspace derivatives. These operators can be either long or short.

Finally, let us consider the interacting quantum theory. The operators which were allowed on analytic superspace classically will split into two groups: those which are subject to a shortening condition which will be analytic tensor fields constructed from derivatives and products of single-trace 1/2 BPS operators, and those which are not short. The latter will not saturate any unitarity bound and this means that there are nearby representations with the same number of components but with anomalous dimensions. Such operators are therefore unprotected and will develop anomalous dimensions. In the quantum theory they will therefore become quasi-tensor superfields on analytic superspace. We note that the protected operators also divide into two classes: there are those which satisfy two unitarity bounds and which, as we have seen, have unique $\mathfrak{gl}(2|2)$ Young tableaux and there are also some operators which satisfy only one unitarity bound. These operators will have one fixed tableau and one which is ambiguous.

It should be noted that there is a subtlety in the precise definitions of the components of some protected operators which can mix with operators in the same classical representation but which are descendants. This was observed in [8] for the case of one-quarter BPS operators and further details were given in [9]. One anticipates that a similar phenomenon should occur for some series B and short series A operators. This complication does not affect the classification we have given above, although one might say that a more precise statement is that there is a one-to-one correspondence between operators which can be written on analytic superspace in terms of derivatives and single-trace 1/2 BPS operators and protected operators in the full quantum theory.

A possible explanation of this behaviour in perturbation theory is as follows: in the free theory the descendant one-quarter BPS operators are short multiplets and so can (and do) mix with operators in the same representations which are not descendants. However, as soon as the coupling is switched on the descendants cease to exist as independent multiplets and become part of irreducible long multiplets. Superconformal symmetry would therefore imply that they cannot mix with the true one-quarter BPS operators. One would therefore expect the mixing to occur only at zeroth order in the coupling; there should not be any quantum corrections. Indeed the quarter BPS operators found in [9] have been shown to remain unmodified at order g^2 [35].

There are also operators such as the Konishi operator which are reducible in the classical interacting theory but which become irreducible in the quantum theory after they acquire anomalous dimensions. In the quantum theory they can therefore be represented by quasi-tensor superfields on analytic superspace. It is easiest to see what happens to these operators in the classical limit by switching off the anomalous dimension at which point we find that the representation becomes reducible.

We now list the different types of gauge-invariant operators in $N = 4$ SYM and give some examples.

3.1.1 CPOs

The simplest protected operators are the single-trace one-half BPS operators (CPOs) which have the form $A_Q := \mathrm{tr}(W^Q)$. These operators are in one-to-one correspondence with Kaluza-Klein states of IIB supergravity on $AdS_5 \times S^5$. This fact was pointed out in [36], although this family of operators had been considered as analytic superfields previously [37]. Indeed, one can explicitly derive the relation between the supergravity and field theory multiplets directly in superspace [38]. The operator $T := A_2$ is special; it is the supercurrent and is extra-short. It has independent components up to fourth order in the odd variables. It also contains all of the conserved currents in the theory. The operator A_3 is also extra-short; it has components up to sixth order in the odd coordinates but contains no conserved currents. All other operators in the sequence are full single component analytic superfields with independent spacetime components up to eighth order in the odd coordinates.

3.1.2 One-half BPS

As well as the CPOs one can also have multi-trace one-half BPS states by taking products of the CPOs. These operators all have Dynkin labels of the form $[000Q000]$ and are therefore given as single-component superfields on analytic superspace. The simplest example is T^2 which has charge 4 and hence is in the same representation as A_4.

3.1.3 One-quarter BPS

The other class of series C operators are the one-quarter BPS states which have Dynkin labels $[00pqp00]$. These are represented by analytic tensor superfields which have p derivatives with both the primed and unprimed indices totally anti-symmetrised. (The Young tableau $\langle 0,0,0,p\rangle$ has p boxes in a single column.) If $p = 1$, these operators are covectors on analytic superspace, and the simplest example of this class is the operator $[0013100]$ which can be written explicitly as

$$\mathcal{O}_{AA'} = \partial_{[A[A'}T\partial_{B]B']}A_3 + \cdots \tag{3.1}$$

where the dots denote further terms required to ensure that the operator is a primary field. Note that these operators, and all other tensor operators, involve derivatives and so must be multi-trace. In the classical theory there are also operators which transform under one-quarter BPS representations which are single-trace. However, these are descendants of long operators and are not BPS in the quantum theory. The simplest example of this behaviour occurs for the representation $[0020200]$. There is a double-trace BPS operator in this representation

$$\mathcal{O}_{ABA'B'} = \partial_{[A[A'}T\partial_{B]B']}T + \cdots . \tag{3.2}$$

There is also a single-trace operator: in the free theory (but with an $\mathrm{SU}(N_c)$ group) this can be written

$$\mathcal{O}_{ABA'B'} = \mathrm{tr}([\partial_{[A[A'}W, W][\partial_{B]B']}W, W]) \tag{3.3}$$

but this expression does not generalise to the interacting theory because of the absence of a covariant derivative on analytic superspace. A systematic study of these operators is given in [8, 9].

3.1.4 Series B and series A protected

A series B operator which saturates the series B unitarity bound has Dynkin labels $[00n_3n_4n_5n_6n_7]$ where $n_6 = n_7+1$ and $n_3 = n_5+n_7+2$ (or $[n_1n_2n_3n_4n_500]$ with $n_2 = n_1+1$ and $n_5 = n_3+n_1+2$). The true (protected) series B operators in this class are at least triple trace, but there can be descendants in the classical theory which are single- or double-trace. The scalar series B operators which saturate the unitarity bound have Dynkin labels of the form $[00(q+2)pq10]$. They have dimension $L = 2q+p+3$ which must be at least six as they are triple-trace. The simplest example has $p = 0$ and $q = 3$, and therefore has $\mathrm{SU}(4)$ labels $[503]$ and $L = 9$. It can be written as

$$\mathcal{O}_{ABCDE,A'B'C'D'E'} = \partial_{AA'}T\partial_{BB'}\partial_{CC'}A_3\partial_{DD'}\partial_{EE'}A_4 + \cdots \tag{3.4}$$

where the unprimed indices are antisymmetrised, and the primed indices are put in the $\langle 0014\rangle$ Young tableau pattern.

A series A protected operator has $n_2 = n_1+1$ and $n_6 = n_7+1$. The simplest example of this type of operator has Dynkin labels $[0102010]$; it is explicitly given by

$$\mathcal{O}_{AB,A'B'} = \partial_{(AA'}T\partial_{B)B'}T + \cdots . \tag{3.5}$$

This is symmetric on both pairs of indices. The internal Dynkin labels are $[020]$ and, since $J_1 = J_2 = 0$, this corresponds to a scalar operator on Minkowksi superspace. It is the square of the supercurrent in the real 20-dimensional representation of $\mathrm{SU}(4)$. The fact that this operator, and some other series A operators, are protected was inferred from correlator results [7, 39, 40, 41]. The simpler explanation in terms of representations was given in [6] where many other examples are discussed. Note that the representation itself does not determine a protected operator of this type; there can be other realisations which have the same quantum numbers in the free theory but which become reducible in the classical interacting theory and long in the quantum interacting theory. For example, the representation $[0102010]$ can also be realised as a product of the Konishi operator and the supercurrent in the free theory. In the classical interacting theory it is reducible, and can no longer be written as a tensor field on analytic superspace because this would require gauge-covariant analytic superspace derivatives acting directly on W. In the full quantum theory this operator transforms under the irreducible representation $[0(1+b)020(1+b)0]$ and has anomalous dimension equal to $2b$.

3.1.5 Semi-protected operators

These are series A operators which saturate one, but not both, unitarity bounds or series B operators which do not saturate the unitarity bound. The simplest series B example has labels [0040020]. The right-hand tableau is a 2×2 square while the left-hand one is four boxes in a column. It can be realised explicitly as

$$\mathcal{O}_{ABCD,A'B'C'D'} = \partial_{AA'}T\partial_{BB'}T\partial_{CC'}\partial_{DD'}T + \cdots \tag{3.6}$$

where the unprimed indices are in the tableau $\langle 0004 \rangle$ (totally antisymmetric) and the unprimed indices are put into the $\langle 0,0,2,2 \rangle$ tableau. It is interesting to observe that the right-hand $\mathfrak{gl}(2|2)$ representation belongs to the series of long representations $\mathcal{R}_{abc}$ described in the next subsection. So, as far as this subgroup is concerned, there are nearby representations with non-integer values of n_2. However, since the left-hand side is in a short representation of the second $\mathfrak{gl}(2|2)$, and since the left and right sides are related because $R = 0$, it follows that n_2 must remain integral in the quantum theory and so the operator will be protected (in the absence of an anomaly).

An example of a series A semi-protected operator is given by [0202210]. It can be realised as

$$\mathcal{O}_{ABCD,A'B'C'D'} = \partial_{AA'}T\partial_{BB'}T\partial_{CC'}\partial_{DD'}A_3 + \cdots \tag{3.7}$$

with the indices projected onto the representations specified by the tableaux $\langle 0022 \rangle$, for the unprimed indices, and $\langle 0013 \rangle$ for the primed indices.

3.1.6 Long operators

The long operators are series A operators for which neither bound is saturated. The simplest example is the Konishi operator. On Minkowski superspace it is an unconstrained scalar superfield with dimension $L = 2+2b$. The Dynkin labels are $[0(1+b)000(1+b)0]$. To view it as a superfield on analytic superspace it is useful to consider first the representation [0200020]. This corresponds to a tensor with the primed and unprimed indices both in the representation with tableau $\langle 0,0,2,2 \rangle$, i.e. a 2×2 square. In appendix we shall show that this gives a scalar superfield on super Minkowski space. Indeed, all of the two-column tableaux $\langle 0,0,k,k \rangle, k \geq 2$ have the same number of components and also correspond to scalar superfields on Minkowski superspace. Moreover, one can take k to be non-integral as long as $k > 1$ without changing the number of components.

There are also long operators which can be represented as tensors on analytic superspace in the classical interacting theory but which become quasi-tensors in the quantum theory. These must be multi-trace. The simplest example is [0200020]. On analytic superspace this is

$$\mathcal{O}_{ABCD,A'B'C'D'} = \partial_{[AA'}\partial_{B]B'}T\partial_{[CC'}\partial_{D]D'}T + \cdots \tag{3.8}$$

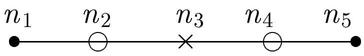
where both sets of indices are projected onto to the 2×2 square tableau representation. In real Minkowski superspace this operator is simply the square of the supercurrent in the singlet representation of SU(4).

3.2 Reducible representations

Any unitary irreducible representation of the $N = 4$ superconformal group can be represented by a Dynkin diagram with labels $[n_1, \ldots, n_7]$, subject to the constraint (2.12), as we have seen. Moreover, one can find how to describe the tensor fields which carry these representations on any coset of the superconformal group with parabolic isotropy group by crossing through some of the nodes. As we have indicated, this procedure specifies the parabolic subgroup and also determines the tensor structure of the field which carries the given representation. In general, such a field may be subject to further differential constraints, as in the case of a scalar field on Minkowski space. However, for analytic superspace this is not the case; the representations are automatically irreducible by holomorphicity. A general analytic superspace is characterised by the property that it only has crosses through internal nodes, whereas harmonic superspaces, Minkowski superspace and super twistor spaces have crosses through the white nodes and/or the external black nodes which correspond to the spacetime spin labels (for the basis choice we have been using). This means that fields on these superspaces may need to satisfy further constraints in order to carry irreducible representations.

For some series A operators it can happen that a superfield in the interacting classical theory describes a reducible, but not completely reducible, representation. This situation corresponds to the existence of descendant operators which transform under short representations in the classical theory but which do not exist as independent representations in the quantum theory due to the presence of anomalous dimensions. That is to say, in the quantum theory, the original long representation becomes irreducible. This problem has recently been studied at great length in [10]; here, we show that it has a very simple description in the analytic superspace formalism. It is related to the notion of quasi-tensors discussed in [19].

To illustrate the phenomenon, consider first the $N = 2$ theory on analytic superspace. The super Dynkin diagram for this space is

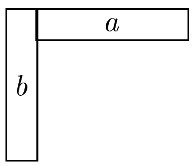


(3.9)

An irreducible representation is specified by the Dynkin labels $[n_1, \ldots, n_5]$. Tensor fields will transform linearly under the two $\mathfrak{sl}(2|1)$ subalgebras. A general $\mathfrak{gl}(2|1)$ Young tableau is specified by three labels $\langle m_1, m_2, m_3\rangle$ and these are related to the (left-hand) $\mathfrak{sl}(2|1)$ Dynkin labels by

$$m_2 - m_1 = n_1 \qquad m_2 + m_3 = n_2 \tag{3.10}$$

so that the $\mathfrak{sl}(2|1)$ representation is left unchanged by adding m to m_1, m_2 and subtracting m from m_3. For representations with $m_1 \neq 0$ it is possible to use this freedom to bring the tableau to the canonical form, $m_1 = 0$. There are two classes of representations, the long, or typical ones,

$\mathcal{R}_{ab} =$ 

(3.11)

corresponding to the $\mathfrak{sl}(2|1)$ labels $[a, a+b]$, and the short, or atypical representations,

$$\mathcal{R}_a = \overbrace{\square\square\cdots\square\square}^{a+1} \tag{3.12}$$

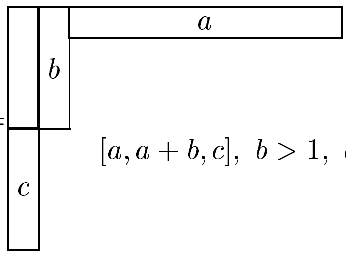
for which the labels are $[a, a+1]$. The latter representations have non-negative, integral values for a, whereas the former have non-negative integral values for a but b can be any real number such that $b > 1$.

If we let $b = 1$ we apparently get $\mathcal{R}_{a1} = \mathcal{R}_a$. However, this is not so. In fact, as one lets b tend to 1, one finds that the representation R_{ab} becomes reducible at $b = 1$, although not completely reducible. We therefore have

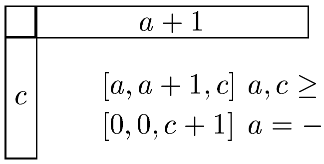
$$\lim_{b\mapsto 1} \mathcal{R}_{ab} \cong \mathcal{R}_a + \mathcal{R}_{a-1}\,. \tag{3.13}$$

This point was discussed explicitly in [19].

Now let us consider the $\mathfrak{gl}(2|2)$ case relevant to $N = 4$ superconformal symmetry. In this case there are again two types of representation which we denote $\mathcal{R}_{abc}$, $\mathcal{R}_{ac}$ with Young tableaux and Dynkin labels

$\mathcal{R}_{abc} =$ 

$[a, a+b, c],\ b > 1,\ a, c \geq 0$ (3.14)

$\mathcal{R}_{ac} =$ 

$[a, a+1, c]\ a, c \geq 0$
$[0, 0, c+1]\ a = -1, c \geq 0$ (3.15)

There is also the trivial representation, $\mathcal{R}_0$, which must be treated separately in this case. For these representations a and c are integral while b can again be non-integral. Again we observe that the limit of $\mathcal{R}_{abc}$ as $b \mapsto 1$ appears to be R_{ac} but this is not so. In fact we find

$$\lim_{b\mapsto 1} \mathcal{R}_{abc} \cong \mathcal{R}_{ac} + \mathcal{R}_{a-1,c+1}\,. \tag{3.16}$$

In terms of Dynkin labels this reads

$$\lim_{b\mapsto 1}[a, a+b, c] \cong [a, a+1, c] + [a-1, a, c+1] \qquad a \geq 1 \tag{3.17}$$
$$\lim_{b\mapsto 1}[0, b, c] \cong [0, 1, c] + [0, 0, c+2] \tag{3.18}$$

As in the $N = 2$ case one can carry out this limiting procedure explicitly. The representations $\mathcal{R}_{abc}$ for fixed a and c all have the same number of components for any real value of $b > 1$. When one takes the limit one finds explicitly that the representation becomes reducible at $b = 1$ according to the pattern we have just described. This can be seen explicitly by taking traces of the representations as was done in [42] (although a different representation of the Young tableaux was used in that paper).

In the field theory context the continuous label b is related to the anomalous dimension of an operator, and the limit $b \mapsto 1$ can be viewed as the classical limit. The reducible representation obtained at the limit can be viewed as the ancestor of the descendant representation given by the smaller of the two representations (the second one) on the right-hand side of equation (3.17) or (3.18).

There are operators which transform under limiting representations for both the left and right $\mathfrak{gl}(2|2)$s. In this case the original representation will split into four in the limit. An example of this is provided by the Konishi operator, $[0(1+b)000(1+b)0]$ which splits into the irreducible representations [0100010], [0100200], [0020010] and [0020200] at $b = 1$. These correspond to the free Konishi operator, two series B descendants and a series C descendant. As noted in [10] series C operators of the form $[001p100]$ cannot arise as descendants, as one can see from (3.18), and so must be protected. It also follows, from (3.17), that series A short representations of the form $[a(a+1)0p0(a+1)a]$ cannot arise as descendants although they can occur as limiting reducible representations. A similar statement holds true for short series B representations of the form $[a(a+1)qp(q+a+2)00]$.

4. Two- and three-point functions

In [19] it was shown how one can solve the $\mathfrak{psl}(4|4)$ Ward identities and write down 2- and 3-point functions of arbitrary protected operators in $N = 4$ SYM. It was also argued that these formulae can be extended to unprotected operators, which can have non-integer dilation weight, if we extend the definition of a tensor superfield to quasi-tensor superfields. The formulae can also be generalised straightforwardly to 4-point functions and higher n-point functions in which case we also have to include functions of invariants. It was also shown that all 2-, 3- and 4-point functions of protected operators are automatically covariant under the bonus $\mathrm{U}_Y(1)$ symmetry. At first sight it appears that this argument might extend to unprotected operators as well. This is not the case as we shall see. In this section we shall only consider representations with (half)-integer dilation weights. In analytic superspace these can all be written as analytic tensor superfields and the point about $\mathrm{U}_Y(1)$ covariance can be illustrated in this case.

We shall be considering solutions to the $\mathfrak{psl}(4|4)$ Ward identities for an n-point correlator,

$$\langle 12\dots n\rangle := \langle \mathcal{O}^{Q_1}_{\mathcal{R}_1\mathcal{R}'_1}(X_1)\dots\mathcal{O}^{Q_n}_{\mathcal{R}_n\mathcal{R}'_n}(X_n)\rangle \tag{4.1}$$

where the operators transform as in (2.24). The Ward identities state that the correlator must be invariant under superconformal transformations. In other words

$$\delta\langle 12\dots n\rangle = \sum_{i=1}^{n}(\mathcal{V}_i + \mathcal{R}_i(A_i) + \mathcal{R}'_i(D_i) + Q_i\Delta_i)\langle 12\dots n\rangle = 0 \tag{4.2}$$

where $A_i := A(X_i)$ $D_i := D(X_i)$. Throughout this section, we shall assume that the Young tableaux corresponding to the representations $\mathcal{R}_i$ are in canonical form. As we have seen, it is always possible to make this choice. A discussion of the $\mathrm{U}_Y(1)$ properties of correlation functions in Minkowski superspace was givne in [44].

4.1 Two-point functions

First we shall consider the 2-point formula given in [19]:

$$\langle \mathcal{O}^Q_{\mathcal{R}\mathcal{R}'}(1)\mathcal{O}^Q_{\mathcal{R}'\mathcal{R}}(2)\rangle \sim (g_{12})^Q \mathcal{R}(X_{12}^{-1})\mathcal{R}'(X_{12}^{-1}) \tag{4.3}$$

where $X_{12} := X_1 - X_2$, and the propagator g_{12} is defined by

$$g_{12} := \mathrm{sdet} X_{12}^{-1} = \frac{\hat{y}_{12}^2}{x_{12}^2} = \frac{y_{12}^2}{\hat{x}_{12}^2} \tag{4.4}$$

where

$$\hat{x}_{12} = x_{12} - \lambda_{12} y_{12}^{-1} \pi_{12} \tag{4.5}$$
$$\hat{y}_{12} = y_{12} - \pi_{12} x_{12}^{-1} \lambda_{12} \tag{4.6}$$

and matrix multiplication is implied, with the inverses having downstairs indices $(x^{-1})_{\alpha'\alpha}$, $(y^{-1})_{a'a}$. In the above formula, one takes I factors of X_{12}^{-1} (I=total number of primed or unprimed indices), with the unprimed indices in the $\mathcal{R}$ representation (in other words, taking symmetric/antisymmetric combinations as dictated by the (canonical) Young tableau for the representation $\mathcal{R}$), this automatically puts the primed indices in the $\mathcal{R}$ representation as well. Similarly, one then takes another I factors of X_{12}^{-1}, with the primed indices in the $\mathcal{R}'$ representation (which also puts the unprimed indices in this representation).

One can check that this satisfies the Ward identities (4.2), given that

$$\begin{aligned} \mathcal{V}X_{12} &= A_1 X_{12} + X_{12} D_2 \\ &= A_2 X_{12} + X_{12} D_1 \end{aligned} \tag{4.7}$$
$$\mathcal{V}g_{12} = -(\Delta_1 + \Delta_2) g_{12}\,. \tag{4.8}$$

The propagator takes care of the $Q_i\Delta_i$ terms, and the Xs take care of the $\mathfrak{sl}(2|2)$ transformations.

In this formula for the 2-point function we have assumed that the Young tableau for $\mathcal{R}$ is in canonical form (2.20). We know, however, that there are other Young tableaux which can specify the same SL(2|2) representation (2.19). We can ask whether they might not be used instead to satisfy the Ward identities. This is indeed true, although in the case of the 2-point function we obtain the same answer as before so the 2-point function is unique. For higher-point functions we can obtain different solutions in this way.

To illustrate this point, consider the replacement of the Young tableau $\mathcal{R} = \langle 0, n_1, n_2 - n_1, n_3 + n_2 - n_1\rangle$ by the Young tableau $\mathcal{S} = \langle m, n_1 + m, n_2 - n_1 - m, n_3 + n_2 - n_1 - m\rangle$ instead of $\mathcal{R}$ which carries the same representation of $\mathfrak{sl}(2|2)$. One can show that

$$\langle \mathcal{O}^Q_{\mathcal{R}\mathcal{R}'}(1)\mathcal{O}^Q_{\mathcal{R}'\mathcal{R}}(2)\rangle \sim (g_{12})^{Q-m} \mathcal{E}(m)^{-1} \mathcal{S}(X_{12}^{-1})\mathcal{E}(m)\mathcal{R}'(X_{12}^{-1}) \tag{4.9}$$

is also a solution to the Ward identities. This is, however, a trivial statement as this expression is in fact equal to the previous expression in (4.3). In fact, if two representation spaces are related by $\mathcal{S} = \mathcal{E}(m)\mathcal{R}$, then we have

$$\mathcal{S}(X_{12}^{-1}) = (\mathrm{sdet}X_{12})^{-m}\mathcal{E}(m)\mathcal{R}(X_{12})\mathcal{E}(m)^{-1} = (g_{12})^m\mathcal{E}(m)\mathcal{R}(X_{12})\mathcal{E}(m)^{-1} \tag{4.10}$$

which is the finite version of equation (2.41), valid as long as $X_{12}^{-1} \in \mathrm{GL}(2|2)$. This formula is easy to check: it is obviously true in the case $X \in \mathrm{SL}(2|2)$, since both $\mathcal{R}$ and $\mathcal{S}$ have the same Dynkin labels, and thus correspond to the same SL(2|2) representation. Clearly both sides of the equation are representations of GL(2|2), so we just have to show they are the same representation. One therefore just needs to check that the equation is true for the matrix $Y = \mathrm{diag}(d, d|d^{-1}, d^{-1})$ since one can obtain all GL(2|2) matrices by multiplying an SL(2|2) matrix by Y. In fact one just needs to check that the highest weight state transforms in the same way under Y, the correct transformation properties of all the other states is guaranteed at the Lie algebra level as we can generate all other states by applying lowering operators from $\mathfrak{sl}(2|2)$.

4.2 Three-point functions

The formula for 3-point functions, given in [19], is

$$\begin{aligned}\langle 123\rangle \sim\ & (g_{12})^{Q_{12}}(g_{23})^{Q_{23}}(g_{31})^{Q_{31}} \times \\ & \mathcal{R}_2(X_{12}^{-1})_{\underline{A}_2\underline{B}_2'}\mathcal{R}_2'(X_{12}^{-1})_{\underline{B}_2\underline{A}_2'}\mathcal{R}_3(X_{13}^{-1})_{\underline{A}_3\underline{B}_3'}\mathcal{R}_3'(X_{13}^{-1})_{\underline{B}_3\underline{A}_3'} \times t(X_{123})_{\underline{A}_1\underline{A}_1'}^{\underline{B}_2\underline{B}_3\underline{B}_2'\underline{B}_3'}\end{aligned} \tag{4.11}$$

where

$$\langle 123\rangle = \langle \mathcal{O}^{Q_1}_{\underline{A}_1\underline{A}_1'}(1)\mathcal{O}^{Q_2}_{\underline{A}_2\underline{A}_2'}(2)\mathcal{O}^{Q_3}_{\underline{A}_3\underline{A}_3'}(3)\rangle \tag{4.12}$$

$$Q_{ij} = \frac{1}{2}(Q_i + Q_j - Q_k)\,, \qquad k \neq i, j \tag{4.13}$$

$$X_{123} = X_{12}X_{23}^{-1}X_{31} \tag{4.14}$$

and the notation $\underline{A}_i$ $i = 1, 2, 3$ represents all the unprimed indices of the operator at the point i, which together carry the representation $\mathcal{R}_i$, and similarly for $\underline{A}_i'$. Then the indices $\underline{B}_i'$ $i = 2, 3$ must also carry the representation $\mathcal{R}_i$ and $\underline{B}_i$ must carry the representation $\mathcal{R}_i'$.

The tensor t is a monomial function of $(X_{123})^{AA'}$, $(X_{123}^{-1})_{A'A}$, δ_A^B and $\delta_{A'}^{B'}$ with index structure as shown. In general there are many different possible such monomials and the complete three-point function will be a linear combination of all the possibilities. We will get further restrictions on the allowed non-vanishing correlation functions due to analyticity in the internal y coordinates which we will consider later.

In contrast to the case of 2-point functions, we can obtain new solutions to the Ward identities at 3 points by using Young tableaux which are not in canonical form. This results in solutions of the same form as (4.13) but with factors of $\mathcal{E}$. Because $\mathcal{E}$ is not $\mathfrak{pgl}(2|2)$ invariant, the powers of the propagators have to be adjusted accordingly.

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4.3 Transformation of correlators under PGL(4|4)

$N=4$ super Yang-Mills is only invariant under PSL(4|4) not PGL(4|4), but we can ask what implications the invariance under PSL(4|4) has for PGL(4|4) transformations. In particular, we can examine the transformation of the correlators, obtained in the previous section by solving the PSL(4|4) Ward identities, under the enlarged group PGL(4|4).

In the examples to be discussed below we shall suppose that all of the operators have canonical tableaux. We shall say that such a correlator has $\mathrm{U}(1)_Y$ charge p if

$$\sum_{i=1\ldots n}(\mathcal{V}_i+\mathcal{R}_i(A_i)+\mathcal{R}'_i(D_i)+Q_i\Delta_i)\langle 1\ldots n\rangle = p\Delta'\langle 1\ldots n\rangle \tag{4.15}$$

where $\mathcal{V}$ defines a $\mathfrak{pgl}(4|4)$ transformation. Note that this definition makes sense even if one cannot assign a definite $\mathrm{U}(1)_Y$ charge to a given operator.

The following two results can be seen relatively straightforwardly:

- All 2-point functions have vanishing $\mathrm{U}_Y(1)$ charge.

  To see this simply consider the expression for the 2-point function $\langle 12\rangle$ (4.3). Since $\mathcal{R}$ and $\mathcal{R}'$ are Young tableaux such that $m_1=m'_1=0$, one finds, under a $\mathfrak{pgl}(4|4)$ transformation, that

  $$(\mathcal{V}_1+\mathcal{V}_2+\mathcal{R}(A_1)+\mathcal{R}'(A_2)+\mathcal{R}'(D_1)+\mathcal{R}(D_2)+Q(\Delta_1+\Delta_2))\langle 12\rangle = 0 \tag{4.16}$$

  where $\mathcal{V}$ is the vector field generating this transformation. This follows because the formulae given in (4.7) and (4.8) remain valid in $\mathfrak{pgl}(4|4)$, as one can easily verify. Thus the 2-point function has vanishing $\mathrm{U}_Y(1)$ charge since the Young tableaux are in the canonical form with $m_1=m'_1=0$ and there is no term involving the $\mathrm{U}_Y(1)$ parameter Δ'.

  Note that this result does not necessarily mean that the 2-point function is PGL(4|4) invariant. If one considers a two-point correlator of the form of an operator and its conjugate, the total $\mathrm{U}(1)_Y$ charge is zero implying PGL(4|4) invariance. However, one can also consider the two-point function of one operator and the conjugate of another which transforms in the same way under PSL(4|4) but which has different $\mathrm{U}(1)_Y$ charge. Since $\mathrm{U}(1)_Y$ is not a symmetry of the interacting theory, there is no reason why such correlators should vanish. In this case, the $\mathrm{U}(1)_Y$ charge of the correlator would still be zero whereas the charge required by PGL(4|4) symmetry would be equal to that of the charged operator. This applies particularly to long operators. One would therefore expect the diagonal combinations not to have well-defined $\mathrm{U}(1)_Y$ charges. In principle, the same reasoning should apply to semi-protected operators.

- All 2-, 3- and 4-point functions of protected operators have vanishing $\mathrm{U}_Y(1)$ charge and are invariant under PGL(4|4). This follows because the Young tableaux for protected operators have $m_1=m'_1=0$ as mentioned at the end of section 2.3. For $n=2,3$ the n-point functions satisfy the equation

  $$\sum_{i=1\ldots n}(\mathcal{V}_i+\mathcal{R}_i(A_i)+\mathcal{R}'_i(D_i)+Q_i\Delta_i)\langle 1\ldots n\rangle = 0 \tag{4.17}$$

where again $\mathcal{V}$ is the vector field generating the $\mathfrak{pgl}(4|4)$ transformation. Since all Young tableaux in this expression must be in the canonical form (i.e. with $m_1 = 0$) this has charge zero. In this case the correlator is PGL(4|4) invariant (i.e. satisfies PGL(4|4) Ward identities $\delta\langle 1 \ldots n >= 0.$) This is because there are no protected operators with non-zero $\mathrm{U}(1)_Y$ charge.

For four or more points, such correlators can be written in terms of explicit functions involving the Xs and the propagators g_{ij} multiplied by functions of the invariants. Because there is no freedom in the choice of tableaux for such operators, the former give rise to $\mathrm{U}(1)_Y$ invariant expressions, and so the $\mathrm{U}(1)_Y$ invariance of n-point functions of protected operators depends on the $\mathrm{U}(1)_Y$ properties of the invariants. As we know, the 4-point invariants are invariant under $\mathrm{U}(1)_Y$ while this is not necessarily true at 5 or more points [18]. This will be discussed in more detail in section 5.

4.4 $\langle LA_{Q_2}A_{Q_3}\rangle$

We denote the chiral primary operators by $A_Q,\ A_Q := \mathrm{tr}(W^Q)$. A 3-point function with one arbitrary operator and two chiral primary operators will have vanishing $\mathrm{U}_Y(1)$ charge as predicted in [22]. From the formula for three point functions (4.11) we obtain

$$\langle \mathcal{O}^{Q_1}_{\mathcal{R}\mathcal{R}} A_{Q_2} A_{Q_3}\rangle \sim \mathcal{R}(X^{-1}_{123}) g_{12}^{Q_{12}} g_{13}^{Q_{13}} g_{23}^{Q_{23}}\,. \tag{4.18}$$

We can again assume that $\mathcal{R}$ is in canonical form, and thus we can see that this expression has zero $\mathrm{U}_Y(1)$ charge. As in the case of two-point functions this does not always mean the correlator is PGL(4|4) invariant, since one could have an operator $\mathcal{O}^{Q,Q'}_{\mathcal{R}\mathcal{R}'}$ on the left-hand side with non-zero $\mathrm{U}_Y(1)$ charge, but the right-hand side would still have zero $\mathrm{U}_Y(1)$ charge.

4.5 A correlation function with non-zero $\mathrm{U}_Y(1)$ charge

We shall now consider examples of correlation functions which do have non-zero $\mathrm{U}_Y(1)$ charge. The simplest example one can construct involves two series C vector operators and one long operator

$$\langle 123\rangle = \langle \mathcal{O}^{Q_1}_{\underline{AA'}} \mathcal{O}^{Q_2}_{BB'} \mathcal{O}^{Q_3}_{CC'}\rangle \tag{4.19}$$

where both sets of indices on the first operator are in the representation $\mathcal{R} = \langle 0033\rangle$, $Q_1 = 6 + d,\ Q_2 = 2 + d_2,\ Q_3 = 2 + d_3$. The ds can be identified with the central Dynkin label of the representations. The operators at points 2 and 3 are such series C operators with internal quantum numbers given by $[1d_i1],\ i = 2, 3$.

The solution (4.13) to the ($\mathfrak{psl}(4|4)$) Ward identities is

$$\langle 123\rangle \sim g_{12}^{Q_{12}} g_{13}^{Q_{13}} g_{23}^{Q_{23}} (X^{-1}_{12})_{B'D} (X^{-1}_{12})_{D'B} (X^{-1}_{13})_{C'E} (X^{-1}_{13})_{E'C}\, t^{DD',EE'}_{\underline{AA'}}(X_{123})\,. \tag{4.20}$$

There are many possible solutions depending on the choice of t. One possibility is

$$t^{DD',EE'}_{\underline{AA'}} = P\left(\delta^D_{A_1}\delta^E_{A_2}\delta^{D'}_{A'_1}\delta^{E'}_{A'_2}\mathcal{S}(X^{-1}_{123})_{\underline{\hat{A}\hat{A}'}}\right) \tag{4.21}$$

where we have written $\underline{A} = (A_1, A_2, \hat{\underline{A}})$ and similarly for the primed indices. The representation $\mathcal{S}$ corresponds to the tableau $\langle 0022\rangle$ and P projects the $\underline{A}$ and $\underline{A}'$ indices onto the tableau $\langle 0033\rangle$. One can check that this is analytic in the internal coordinates if

$$d + d_2 - d_3 \geq 2 \tag{4.22}$$

$$d + d_3 - d_2 \geq 2 \tag{4.23}$$

$$d_2 + d_3 - d \geq 4\,. \tag{4.24}$$

Under a $\mathfrak{pgl}(4|4)$ transformation this solution satisfies the equation

$$\sum_{i=1\ldots 3} (\mathcal{V}_i + \mathcal{R}_i(A_i) + \mathcal{R}'_i(D_i) + Q_i\Delta_i)\langle 123\rangle = 0 \tag{4.25}$$

and therefore has vanishing $\mathrm{U}_Y(1)$ charge.

However, we can also exhibit a solution involving the $\mathcal{E}$-tensor. It is

$$\langle 123\rangle \sim g_{12}^{Q_{12}-\frac{1}{2}} g_{13}^{Q_{13}-\frac{1}{2}} g_{23}^{Q_{23}+\frac{1}{2}} (X_{12}^{-1})_{B'D}(X_{12}^{-1})_{D'B}(X_{13}^{-1})_{C'E}(X_{13}^{-1})_{E'C}\tilde{t}_{\underline{AA'}}^{DD',EE'}(X_{123}) \tag{4.26}$$

where

$$\tilde{t}_{\underline{AA'}}^{DD',EE'} = \mathcal{E}^{-1}\tilde{P}\left(\delta_{A_1}^{D}\delta_{A_2}^{E}\delta_{A'_1}^{D'}\delta_{A'2}^{E'}\mathcal{S}(X_{123}^{-1})_{\underline{\hat{A}\hat{A}'}}\right) \tag{4.27}$$

where $\tilde{P}$ projects onto $\langle 0033\rangle$ on the unprimed covariant indices and onto $\langle 1122\rangle$ on the primed indices. The $\mathcal{E}^{-1}$ then acts on the primed indices to bring them back into the $\langle 0033\rangle$ pattern.

This solution is valid (i.e. analytic in the internal coordinates) when

$$d + d_2 - d_3 \geq 1 \tag{4.28}$$

$$d + d_3 - d_2 \geq 1 \tag{4.29}$$

$$d_2 + d_3 - d \geq 1\,. \tag{4.30}$$

Under a PGL(4|4) transformation this solution satisfies

$$\sum_{i=1\ldots 3} (\mathcal{V}_i + \mathcal{R}_i(A_i) + \mathcal{R}'_i(D_i) + Q_i\Delta_i)\langle 123\rangle = \Delta'\langle 123\rangle\,. \tag{4.31}$$

This is again a solution of the superconformal Ward identities (since for PSL(4|4) transformations $\Delta' = 0$) but it has non-vanishing $\mathrm{U}_Y(1)$ charge.

One would expect a general 3-point function of this type to be given by a linear combination of the above solutions (and other possible solutions) so that the 3-point function of such operators will not have a well-defined $\mathrm{U}_Y(1)$ charge.

This example can be generalised to more complicated three-point functions of the same type, i.e one-quarter BPS series C operators at points 2 and 3 and a long operator at point 1 with suitably chosen representations. The construction can be further extended in two ways: the long operator can be allowed to be a quasi-tensor (which is the interesting case in practice), and it can be replaced by a semi-protected operator. In all of these cases the basic principle is the same: one can find two (or more) different solutions of the Ward Identities involving two (or more) different representations of $\mathfrak{gl}(2|2)$ which are equivalent in $\mathfrak{sl}(2|2)$.

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4.6 Semi-protected operators

The proof that 3- and 4-point functions involving protected operators are $\mathrm{U}_Y(1)$ invariant depended on both the left and right GL(2|2) representations of both operators being short and therefore does not apply to series B operators which do not saturate the series B bound or to series A operators which only saturate one unitary bound. This is because for these operators, although one GL(2|2) representation is short, the other is long. We call such operators semi-protected; even though the operators themselves are non-renormalised [6], 2- and 3-point functions involving them are in general not $\mathrm{U}_Y(1)$ invariant. Despite this, however, it is still possible to show that the 2-point functions of semi-protected operators are non-renormalised using a slight generalisation of the usual argument.

The non-renormalisation of the two-point functions of semi-protected operators is proved using the reduction formula which relates the derivative of the two-point function with respect to the complex coupling τ to the three point function involving an insertion of the energy-momentum supercurrent:

$$\frac{\partial}{\partial\tau}\langle\mathcal{O}\bar{\mathcal{O}}\rangle \sim \int \mathrm{d}\mu_0\langle T_0\mathcal{O}\bar{\mathcal{O}}\rangle \tag{4.32}$$

where $\mathrm{d}\mu_0 = \mathrm{d}^4x\mathrm{d}^4\lambda\mathrm{d}^4y$. It will be important to note that the measure has $\mathrm{U}_Y(1)$ charge $+2$.

We now show that the correlation function involving T, one semi-protected operator, $\mathcal{O}$, and its conjugate operator, $\bar{\mathcal{O}}$,

$$\langle T\mathcal{O}^Q_{\underline{AB'}}\bar{\mathcal{O}}^Q_{\underline{BA'}}\rangle \tag{4.33}$$

can have $\mathrm{U}_Y(1)$ charge at most 1 so the right-hand side of (4.32) vanishes. Note that previous non-renormalisation theorems relied on the fact that the correlation functions were $\mathrm{U}_Y(1)$ invariant, i.e. had charge zero. Here the indices $\underline{A}$ and $\underline{A}'$ are in the same short representation $\mathcal{R}$ which has an L-shaped Young tableau $\langle 0, m_2, 1, m_4\rangle$, while the $\underline{B}$ and $\underline{B}'$ indices are in an arbitrary (long) representation $\mathcal{S}$ which we assume to be canonical ($m_1 = 0$).

Before writing down the three-point function, we make a small digression on the three-point function of two vector operators and one T. From the general formula (4.11) we have

$$\langle T\mathcal{O}_{AB'}\mathcal{O}_{BA'}\rangle \sim (X_{12}^{-1})_{C'A}(X_{12}^{-1})_{B'D}(X_{13}^{-1})_{D'B}(X_{13}^{-1})_{A'C}t(X_{123})^{CC',DD'} \tag{4.34}$$

where we have omitted the propagator factors. There are two possible choices for t:

$$(a)\quad t^{CC',DD'} = (X_{123})^{CC'}(X_{123})^{DD'} \tag{4.35}$$

$$(b)\quad t^{CC',DD'} = (X_{123})^{CD'}(X_{123})^{DC'}\,. \tag{4.36}$$

These lead to the following solutions for the three-point function:

$$(a)\quad \langle T\mathcal{O}_{AB'}\mathcal{O}_{BA'}\rangle \sim (X_{23}^{-1})_{A'A}(X_{23}^{-1})_{B'B} \tag{4.37}$$

$$(b)\quad \langle T\mathcal{O}_{AB'}\mathcal{O}_{BA'}\rangle \sim (X_{312}^{-1})_{A'B}(X_{231}^{-1})_{B'A}\,. \tag{4.38}$$

The idea now is to rewrite the three-point function (4.33) making use of these two basic solutions. That is, we factorise the $\mathcal{R}$ and $\mathcal{S}$ representations, and write a solution of type (a) for the first factor multiplied by a solution of type (b) for the second factor. In addition, in order to obtain a non-zero $\mathrm{U}(1)_Y$ charge, we shall have to change the primed (or unprimed) long representation $\mathcal{S}$ to a non-canonical one, $\tilde{\mathcal{S}}$, with $m_1' = m$, say. We then have to supply an appropriate factor of $\mathcal{E}$ to return the indices into the same representations as the left-hand side. A solution of this form is

$$\begin{aligned}\langle T\mathcal{O}^Q_{\underline{AB'}}\bar{\mathcal{O}}^Q_{\underline{BA'}}\rangle \sim\ & g_{12}g_{13}g_{23}^{Q-1-m}\mathcal{E}(m)^{-1}\times \\ & \times P\left(\mathcal{R}_1(X_{23}^{-1})_{\underline{A}_1\underline{A}_1'}\ \mathcal{R}_2(X_{23}^{-1})_{\underline{B}_1\underline{B}_1'}\ \mathcal{R}_3(X_{213}^{-1})_{\underline{B}_2\underline{A}_2'}\ \mathcal{R}_4(X_{312}^{-1})_{\underline{A}_2\underline{B}_2'}\right),\end{aligned} \tag{4.39}$$

where we have split the multi-indices into two sets $\underline{A} \to (\underline{A}_1, \underline{A}_2)$ etc. The operator P projects the $\underline{A}$ and $\underline{A}'$ indices onto the representation $\mathcal{R}$, the $\underline{B}$ indices onto the representation $\mathcal{S}$ and the $\underline{B}'$ indices onto the $\tilde{\mathcal{S}}$ representation. Finally, the $\mathcal{E}$ tensor acts on the $\underline{B}'$ indices to bring them back into the representation $\mathcal{S}$. The $\tilde{\mathcal{S}}$ Young tableau is in non-canonical form with $m_1' = m$. It defines the same $\mathrm{SL}(2|2)$ representation as $\mathcal{S}$ but they differ in $\mathrm{GL}(2|2)$. Such a solution, if it exists, will have $\mathrm{U}(1)_Y$ charge m.

For the projection operator P to give a non-zero result we clearly require that $\mathcal{R}$ is contained in the tensor product $\mathcal{R}_1 \otimes \mathcal{R}_3$, and similarly for the other indices. So we must have

$$\mathcal{R}_1 \otimes \mathcal{R}_3 \ni \mathcal{R} \tag{4.40}$$

$$\mathcal{R}_1 \otimes \mathcal{R}_4 \ni \mathcal{R} \tag{4.41}$$

$$\mathcal{R}_2 \otimes \mathcal{R}_3 \ni \mathcal{S} \tag{4.42}$$

$$\mathcal{R}_2 \otimes \mathcal{R}_4 \ni \tilde{\mathcal{S}}\,. \tag{4.43}$$

Now clearly a long representation multiplied by any other representation will still be long. Therefore, since $\mathcal{R}$ is short (4.40) and (4.41) imply that $\mathcal{R}_1$, $\mathcal{R}_3$ and $\mathcal{R}_4$ are all short representations. Similarly, a representation in canonical form (with $m_1 = 0$) cannot be obtained in the tensor product of a representation with $m_1 \neq 0$ and any other representation (this can be seen by considering the multiplication of Young tableaux). Since $\mathcal{S}$ is in canonical form, by assumption, (4.42) implies that $\mathcal{R}_2$ must be in canonical form ($m_1 = 0$). Finally, (4.43) tells us that $\tilde{\mathcal{S}}$ is contained in the tensor product of a Young tableau in canonical form ($\mathcal{R}_2$) and a short representation($\mathcal{R}_4$).

For a semi-protected operator in series A, the Young tableau for $\mathcal{R}$ has the form $\langle 0, k, 1, q+2r+s-(k+1)\rangle$ while the Young tableau for $\mathcal{S}$ has the form $\langle 0, s, r, r+q\rangle$. Now let us suppose that $\tilde{\mathcal{S}}$ is $\langle 2, s+2, r-2, (r+q)-2\rangle$, so that $m = 2$, and that the tableau for $\mathcal{R}_2$ is $\langle 0, p_2, p_3, p_4\rangle$. When $\mathcal{R}_2$ is multiplied by any representation, the third and fourth tableau numbers cannot be diminished, so that, at most, $p_3 = r-2$ and $p_4 = (r+q)-2$. Suppose this is the case, then since the two left-most columns of $\mathcal{S}$ and $\tilde{\mathcal{S}}$ differ by a subtableau in the form of a 2×2 square, it follows that the $\mathcal{R}_3$ factor of the short representation $\mathcal{R}$ should contain such a subtableau. But this is impossible since $\mathcal{R}$ is a short tableau in the form of an L (with arms which both have single-box width). Clearly the situation is not improved by reducing p_3 and/or p_4, so we conclude that there is no solution with $m = 2$. The situation is even more constrained as we increase m, whereas it is possible to obtain $m = 1$. We therefore conclude that the maximum value of the $\mathrm{U}(1)_Y$ charge of the three-point correlator (4.33) is 1. This holds in series A, but it is also true in series B. In fact, the result is easier to see in this case because $\mathcal{R}$ only has $m_4 \neq 0$, and this certainly cannot have a subtableau in the form of a 2×2 square.

In conjunction with the reduction formula this result implies that two-point functions of semi-protected operators are non-renormalised since the measure on the right-hand side of the reduction formula has $\mathrm{U}(1)_Y$ charge -2.

Such arguments do not seem to extend to the case of three-point functions, however, and we expect three-point functions involving semi-protected operators to receive quantum corrections in general. Indeed, as we have remarked, three-point functions with one semi-protected operator and two series C operators will in general violate $\mathrm{U}(1)_Y$ symmetry.

4.7 Comparison with earlier results

In reference [22] it was conjectured that three-point functions of short operators and three-point functions with one long and two short operators should obey the $\mathrm{U}(1)_Y$ selection rule, i.e. that such correlators would vanish unless the sum of the $\mathrm{U}(1)_Y$ charges is zero. As we have seen, in the interacting theory, it is not clear that a general long or semi-protected operator has a well-defined $\mathrm{U}(1)_Y$ charge, but we can nevertheless rephrase the conjecture in terms of the $\mathrm{U}(1)_Y$ charges of the correlators which we have defined for the case when all the operators involved have canonical Young tableaux. In addition, short operators were taken to be CPOs in [22], i.e. single-trace one-half BPS operators, and long operators were also taken to be single-trace. We can therefore update these earlier results by replacing the short operators of [22] by arbitrary protected operators, and by allowing semi-protected operators and arbitrary long operators.

For the two-point functions we agree with [22] in the sense that, as we have seen, the $\mathrm{U}(1)_Y$ charge of any two-point function must vanish. However, this does not necessarily imply $\mathrm{U}(1)_Y$ symmetry of such correlators because of the fact that this group is not a symmetry of the interacting theory. As we have mentioned, one can therefore have non-vanishing correlators of two operators which transform under conjugate representations of $\mathfrak{psl}(4|4)$ but which have different $\mathrm{U}(1)_Y$ charges. The diagonalisation of sets of long or semi-protected operators with the same $\mathfrak{psl}(4|4)$ quantum numbers would therefore be expected to yield orthogonal operators with ill-defined $\mathrm{U}(1)_Y$ charges.

For three-point functions of short operators we agree with [22]. Indeed, all protected operators have zero $\mathrm{U}(1)_Y$ charges (as superfields; the higher components within a supermultiplet will be charged), and all two- and three-point functions of such operators also have vanishing $\mathrm{U}(1)_Y$ charge and are invariant under $\mathfrak{pgl}(4|4)$ [19].

Three-point functions with two CPOs or, more generally, two one-half BPS operators and one long or semi-protected operator will have vanishing $\mathrm{U}(1)_Y$ charge as we have seen, although this does not necessarily imply $\mathrm{U}(1)_Y$ invariance. This is in agreement with [22] in the sense that the charge of the correlator vanishes. On the other hand, three-point functions of two more general protected operators and one long or semi-protected operator need not have vanishing $\mathrm{U}(1)_Y$ charge. As we have shown, there are three-point functions with two series C one-quarter BPS operators for which one can explicitly exhibit solutions with zero and non-zero $\mathrm{U}(1)_Y$ charges indicating that the charge of the correlator itself is not well-defined.

5. Superconformal invariants in analytic superspace

In this section we shall consider the construction of rational superconformal invariants (under the group PSL(4|4)) of n points in analytic superspace. These form a ring $\mathcal{I}$ which has a nilpotent ideal $\mathcal{N}$. All of the elements of the quotient ring $\mathcal{Q} := \mathcal{I}/\mathcal{N}$ were given in [23], but it has subsequently become apparent that there are indeed nilpotent invariants [18] starting at $n = 5$ points. In fact, as pointed out in [11], elements of $\mathcal{Q}$ are invariant under $\mathrm{U}(1)_Y$ and hence under the larger group PGL(4|4). The fact that there are no nilpotent invariants for $n \leq 4$ is one way of interpreting the non-renormalisation theorems for 2- and 3-point correlators of protected operators. In reference [29] a sketch was given of how one might construct the invariants completely. Here, we complete the procedure, at least in principle, and also show how to detect the presence of non-$\mathrm{U}(1)_Y$ invariants in a straightforward way.

5.1 Coordinate approach

Consider the transformation

$$\delta X_i = B + AX_i + X_i D + X_i C X_i\,, \qquad i = 1, \ldots, n\,, \qquad \text{no sum on i} \tag{5.1}$$

where X_i is the supercoordinate $X_i^{AA'}$ of the ith point. We are looking for functions $F(X_1, \ldots, X_n)$ which are invariant under the above. We first solve for translations B. If we change coordinates to $(X_1, X_{1i}), i = 2, \ldots, n$ we find that F is independent of X_1. Now consider the transformation of X_{1i} under C,

$$\delta_C X_{1i} = X_1 C X_1 - X_i C X_i - X_{1i} C X_{1i} + X_1 C X_{1i} + X_{1i} C X_1\,. \tag{5.2}$$

For the inverse we therefore have

$$\delta_C X_{1i}^{-1} = C - (CX_1) X_{1i}^{-1} - X_{1i}^{-1} (X_1 C)\,. \tag{5.3}$$

At this stage we can regard F as being a function of the $n-1$ inverses X_{1i}^{-1} and change variables to X_{12}^{-1} and $n-2$ variables Y_i defined by

$$Y_i^{-1} := X_{1i}^{-1} - X_{12}^{-1}\,, \qquad i = 3, \ldots, n\,. \tag{5.4}$$

We note, for future reference, that

$$Y_i = X_{12} X_{2i}^{-1} X_{i1} := X_{12i}\,. \tag{5.5}$$

We have

$$\delta_C Y_i = Y_i (CX_1) + (X_1 C) Y_i \tag{5.6}$$

and so the invariance of F under C implies

$$\delta_C F = \left(C - (CX_1)X_{12}^{-1} - X_{12}^{-1}(X_1C)\right)\frac{\partial F}{\partial X_{12}^{-1}} + \sum_{i=3}^{n}(Y_i(CX_1) + (X_1C)Y_i)\frac{\partial F}{\partial Y_i} = 0\,. \quad (5.7)$$

Now F is independent of X_1 so the above should be valid for arbitrary values of this coordinate. Taking $X_1 = 0$ we see that F does not depend on X_{12}^{-1}. Thus F depends only on the $(n-2)$ Y_i's and the residual transformation reduces to linear transformations of type A and D. Hence, if F, as a function of the Y_i, is invariant under the linear A and D symmetries it will automatically be completely invariant.

The A and D transformations are

$$\delta Y_i = AY_i + Y_iD \quad (5.8)$$

so if we change variables again to Y_3 and $Z_i := Y_iY_3^{-1}$, $i = 4, \ldots, n$, we find that the Zs do not transform under D, only under A:

$$\delta Z_i = AZ_i - Z_iA\,. \quad (5.9)$$

The Z_is can also be written as

$$Z_i := X_{12}\left(X_{2i}^{-1}X_{i1}X_{i3}^{-1}X_{32}\right)X_{12}^{-1}\,. \quad (5.10)$$

The function $F(Y_3, Z_i)$ will then be an invariant if

$$\left((AY_3 + Y_3D)\frac{\partial}{\partial Y_3} + \sum_{i=1}^{n}(AZ_i - Z_iA)\frac{\partial}{\partial Z_i}\right)F = 0\,. \quad (5.11)$$

Now in $N = 2$ analytic superspace we can easily solve for invariance under D transformations. This superspace is defined in a similar way to $N = 4$ analytic superspace but now the internal indices a, a' only take on one value each. The matrices A and D are $\mathfrak{gl}(2|1)$ matrices with $\mathrm{str}A = \mathrm{str}D$. This means we can take A to be straceless and use D to conclude that F must be independent of Y_3. So invariants in $N = 2$ analytic superspace are functions of $(n-3)$ variables Z_i which are $(2|1) \times (2|1)$ supermatrices and which transform under the adjoint representation of $\mathfrak{sl}(2|1)$.

In $N = 4$ the situation is a little more complicated because we cannot remove the straces from the matrices A and D in an invariant manner. However, a similar result can be obtained after some redefinitions. By using all of the parameters in D except for the supertrace, one can show that F depends only on the Zs and $W := \mathrm{str}Y_3$. The invariance condition is now

$$\sum_i [A, Z_i]\frac{\partial F}{\partial Z_i} + \mathrm{str}(A+D)W\frac{\partial F}{\partial W} = 0\,. \quad (5.12)$$

If we replace Z_i by

$$Z_i' := MZ_iM^{-1} \quad (5.13)$$

where

$$M = \begin{pmatrix} W^{-\frac{1}{8}} & 0 \\ 0 & W^{\frac{1}{8}} \end{pmatrix} \quad (5.14)$$

and where each entry in M is a 2×2 matrix, we find

$$\delta Z'_i = [A', Z'_i]\,. \tag{5.15}$$

Writing the parameter matrix in 2×2 block form as

$$A = \begin{pmatrix} A_0 & \Gamma \\ \Upsilon & A_1 \end{pmatrix} \tag{5.16}$$

where A_0, A_1 are even and Γ, Δ are odd, we find

$$A' = \begin{pmatrix} A'_0 & \Gamma' \\ \Upsilon' & A'_1 \end{pmatrix} \tag{5.17}$$

where

$$\begin{aligned} A'_0 &= A_0 - \frac{1}{8}\mathrm{str}(A+D) \\ A'_1 &= A_1 + \frac{1}{8}\mathrm{str}(A+D) \\ \Gamma' &= W^{-\frac{1}{4}}\Gamma \\ \Upsilon' &= W^{+\frac{1}{4}}\Upsilon\,. \end{aligned} \tag{5.18}$$

Note that $\mathrm{str}A' = \frac{1}{2}\mathrm{str}(A-D) = \Delta'$, the $\mathrm{U}(1)_Y$ parameter. So we can regard A' as a general $\mathfrak{gl}(2|2)$ matrix. If we now drop the primes on both the Zs and A we find that an n-point analytic superspace invariant F is a function of $n-3$ $(2|2) \times (2|2)$ supermatrix variables Z_i subject to the differential constraint

$$\sum_{i=3}^{n} [A, Z_i] \frac{\partial F}{\partial Z_i} = 0\,. \tag{5.19}$$

The finite version of the transformation of the Zs is

$$Z_i \mapsto G Z_i G^{-1} \tag{5.20}$$

where G can now be regarded as an element of $\mathrm{GL}(4|4)$.

Note that the unit matrix in A drops out of (5.19), so that this equation expresses invariance under the adjoint action of $\mathfrak{pgl}(2|2)$ on the Z variables. Furthermore, if we restrict A to satisfy $\mathrm{str}A = 0$, then F will be invariant under $\mathrm{PSL}(4|4)$; however, if F is invariant under unrestricted A transformations then it will be invariant under $\mathrm{PGL}(4|4)$. So this approach gives a very simple way of distinguishing the two types of invariant. Alternatively, if we use the finite version, then (5.20) defines a $\mathrm{PGL}(2|2)$ transformation; if F is invariant under transformations of the Zs for which $\mathrm{sdet}G = 1$ it will be $\mathrm{PSL}(4|4)$ invariant, whereas if G is unrestricted it will be invariant under $\mathrm{PGL}(4|4)$.

The final step in the construction of invariants is to solve equation (5.19) (or (5.20)). One way of doing this is to look for invariant polynomials in the components of the Zs. For a kth degree moynomial this is equivalent to looking for invariant tensors of k covariant and

k contravariant $\mathbb{C}^{2|2}$ vectors. The simplest such invariants are given by taking supertraces of the Zs and these coincide with the invariants found in [23]. However, for a sufficiently large number of indices, one can use the $\mathcal{E}$-tensor which is invariant under $\mathfrak{sl}(2|2)$ but not under $\mathfrak{gl}(2|2)$ - these invariants will clearly not be invariant under $\mathrm{U}(1)_Y$. This tensor has twelve indices and so requires a sixth degree monomial in the Zs. The six covariant indices should be projected onto the tableau $\langle 0033 \rangle$, while the upper indices should be put in the $\langle 1122 \rangle$ tableau arrangement. Clearly, this cannot be done with only one Z, and so the lowest number of points at which one can construct a non-$\mathrm{U}(1)_Y$ invariant is therefore $n = 5$ as we know from other arguments.

In summary, approaching the problem in this way, the n-point invariants are formed from homogeneous polynomials of the $(n-3)$ Z variables by suitably contracting the indices with δs or the invariant tensor $\mathcal{E}$. Those which involve the latter will not be invariant under $\mathrm{U}(1)_Y$.

An alternative way to carry out the last step is to use up the remaining parameters to reduce the number of components contained in the Zs. Let us consider first the simpler case of $n = 4$ points in ordinary Minkowski space. We can apply a very similar analysis to this case and conclude that the n-point invariants are given by $(n-3)$ 2×2 matrix variables z_i which transform at the final step by $z_i \mapsto g z_i g^{-1}$ where $g \in \mathrm{SL}(2)$. For $n = 4$ we only have one z variable, and so one way of constructing the invariants would be to form traces of products of zs. Only two of these are functionally independent, so that there are two 4-point invariants as expected. Another approach is to use the g transformation to bring z to a diagonal form, $z = \mathrm{diag}(x_1, x_2)$. This form is invariant under infinitesimal A-transformations for which A is diagonal and traceless. So the two invariants can be replaced by the eigenvalues x_1, x_2. For more than four points we can use the residual g transformation to remove one more variable leaving us with $4n - 15$ independent invariants for $n \geq 5$. We remark that the variables x_1, x_2 are those used in ref [43] in their discussion of 4-point functions (and denoted x, z by the authors).

Now let us return to the supersymmetric case. Again we start with $n = 4$ points, and only one Z. By a finite transformation of the form $Z \mapsto GZG^{-1}$, $G \in \mathrm{GL}(2|2)$ we can bring Z to a block-diagonal form. In other words, we can eliminate all of the odd variables in Z. Moreover, it is in principle possible to find the matrix G which effects this change. We can then diagonalise the two 2×2 even submatrices to make Z diagonal, $Z = \mathrm{diag}(X_1, X_2 | Y_1, Y_2)$. This means that there are 4 independent superinvariants at $n = 4$ points, and they are all invariant under $\mathrm{U}(1)_Y$. They can also be written as $\mathrm{str}(Z^p)$, $p = 1, \ldots, 4$. At 5 or more points, we can eliminate the odd variables of one Z, Z_4 say, as before. There is a residual (infinitesimal) symmetry consisting of diagonal matrices A which corresponds to only 3 parameters as the unit matrix does not act. We can use two of these parameters to bring the even 2×2 submatrices of Z_5, say, to triangular form, and this form is preserved by As of the form $A = \mathrm{diag}(aI_2, bI_2)$. So the only residual transformation is in fact $\mathrm{U}(1)_Y$ which acts only on the odd coordinates, half with positive charge and half with negative charge. At this point it looks as if we can construct odd invariants as well as even ones, but these are in fact not rational because of the fractional powers of W used above in the redefinitions of the Zs. The redefinition (5.13) affects only the odd coordinates and requires them to occur in particular combinations. In fact, if we write

$$Z = \begin{pmatrix} z & \theta \\ \phi & w \end{pmatrix} \tag{5.21}$$

then each invariant must involve the odd variables in the form $\theta^p \phi^q$ where $p - q = 0$ mod 4. This is another way of spotting $\mathrm{U}(1)_Y$ invariance, since those invariants which do have this symmetry will have $p = q$ in each contributing term.

5.2 Grassmannian approach

In reference [23] $N = 4$ superconformal invariants in analytic superspace were discussed in a slightly different formalism. We briefly review this and then use the $\mathcal{E}$ tensor to construct the missing invariants, not invariant under PGL(4|4). We first consider the space $\mathbb{U}$ of $(2|2) \times (4|4)$ matrices of maximal rank. This space is acted on in a natural way by GL(2|2) from the left and GL(4|4) from the right. We can then form the quotient space $\mathrm{GL}(2|2)\backslash\mathbb{U}$. This can easily be identified with the Grassmannian of (2|2) planes in $\mathbb{C}^{4|4}$, and is another description of analytic superspace. It is straightforward to see that only PGL(4|4) acts on analytic superspace.

We shall now consider the construction of n-point invariants on this space. With each point i we associate $u_i \in \mathbb{U}$. An invariant F is then a function of the us which is invariant under each $\mathrm{GL}(2|2)_i$ (one for every point) separately, and under $u_i \mapsto u_i g,\ g \in \mathrm{PGL}(4|4)$ jointly. We can single out two points, 1 and 2, say, and form the $(4|4) \times (4|4)$ matrix u_{12} formed by taking the upper rows to be given by u_1 and the lower rows to be given by u_2. This matrix transforms from the left by $h_{12} := \mathrm{diag}(h_1, h_2),\ h_i \in \mathrm{GL}(2|2)_i$. We now define two families of $(2|2) \times (2|2)$ matrices by

$$\begin{aligned} K_i &:= u_i (u_{12}^{-1})^1, \qquad i = 3, \ldots, n \\ L_i &:= u_i (u_{12}^{-1})^2, \qquad i = 3, \ldots, n \end{aligned}$$

where the numerical superscripts on the inverse matrix denote the projections onto the corresponding submatrices. These matrices are invariant under GL(4|4) and transform by

$$\begin{aligned} K_i &\mapsto h_i K_i h_1^{-1} \\ L_i &\mapsto h_i K_i h_2^{-1}\,. \end{aligned}$$

Set

$$M_i = (K_i)^{-1} L_i, \qquad i = 3, \ldots, n \tag{5.22}$$

to obtain $(n-2)$ matrices which transform only under h_1 and h_2,

$$M_i \mapsto h_1 M_i h_2^{-1}\,. \tag{5.23}$$

The matrices

$$N_i = M_i M_3^{-1}, \qquad i = 4, \ldots, n \tag{5.24}$$

are invariant under h_2 and transform under the adjoint representation of $\mathrm{GL}(2|2)_1$,

$$N_i \mapsto h_1 N_i h_1^{-1}\,. \tag{5.25}$$

GL(2|2) invariants constructed from the $n-4$ Ns will then be invariant under PGL(4|4). These invariants are the supertraces, or, to put it another way, one can consider a polynomial in powers of the matrix elements of the Ns and hook the indices together with the invariant matrix δ. These were the invariants identified in [23]. They are non-nilpotent and in fact determine the elements of $\mathcal{Q}$. However, as we now know, one can also form SL(2|2) invariants using the $\mathcal{E}$ tensor. The simplest one requires at least two Ns and so occurs at five points. At first sight it might seem that such objects are not actually defined on the quotient space because they are only invariant under SL(2|2) and not under GL(2|2), but this can easily be remedied. If we define

$$W := \operatorname{sdet} u_{12} \operatorname{sdet} M_3 \tag{5.26}$$

then W transforms by

$$W \to (\operatorname{sdet} h_1)^2 \operatorname{sdet} g\, W\,. \tag{5.27}$$

Now any SL(2|2) invariant function of the $\{N_i\}$, not invariant under GL(2|2), will transform by some power p of the superdeterminant of h_1, and this can be compensated for by multiplying it by $W^{-\frac{p}{2}}$. The resulting object will be defined on the coset and is invariant under PSL(4|4) but will clearly not be invariant under PGL(4|4). These are the superconformal invariants which are not invariant under the $\mathrm{U}(1)_Y$ bonus symmetry.

It is easy to compare this formalism with the coordinate approach. One can fix the local GL(2|2) symmetries by choosing representatives of the form

$$X \mapsto s(X) = (1,\ X) \in \mathbb{U} \tag{5.28}$$

at each point. One can then convert the invariants from the Grassmannian formalism to those of the coordinate approach.

5.3 Application

We are now in a position to solve the Ward Identities for n-point correlation functions of arbitrary operators in the $N=4$ superconformal field theory in terms of propagators, known functions of the coordinates and arbitrary functions of the invariants which we have just seen how to construct. The propagator factor absorbs the dilation weights of the operators; in general, it is not uniquely determined, but any two admissible propagator factors will be related to one another by an invariant function. As in the case of the three-point function, each point $j, j \neq 1$, can be translated to point 1, say, by using X_{1j}^{-1} in the appropriate representation. One then absorbs the free indices by tensors t which are monomials in the variables $X_{12k} = X_{12} X_{2k}^{-1} X_{k1},\ k = 3, \ldots, n$ and their inverses. The t-tensors may also involve δs and $\mathcal{E}$s, and the presence of the latter requires the propagator factor to be adjusted appropriately. In general, there will be many possible ts and each independent one can be multiplied by an arbitrary function of the n-point invariants and the coupling constant. Any such solution will then be subject to the constraint that it should be analytic in the internal y coordinates. This prescription solves the superconformal Ward

Identities for all components of all correlators of arbitrary gauge-invariant operators in the theory. The schematic form of the solution is

$$\langle 12\ldots n\rangle = \Pi_{j=2}^{n}\mathcal{R}_j(X_{1j}^{-1})\mathcal{R}'_j(X_{1j}^{-1})\sum_t t_{\mathcal{R}_1\mathcal{R}'_1}^{\mathcal{R}_2\ldots\mathcal{R}_n;\mathcal{R}'_2\ldots\mathcal{R}'_n} P_t F_t \tag{5.29}$$

where the sum is over the possible tensors t, P_t denotes an appropriate propagator factor, which will depend on t if the latter involves factors of $\mathcal{E}$, while $\mathcal{R}_i$ and $\mathcal{R}'_i, i = 1,\ldots,n$, are the representations of the primed and unprimed $\mathfrak{gl}(2|2)$ algebras under which the operators transform. Each F_t is an arbitrary function of the invariants and the coupling which is restricted by analyticity. The formula makes sense if the correlator includes long operators because the objects $\mathcal{R}(X)$ can be defined for quasi-tensors [19]. Moreover, non-integral powers on the internal coordinates are cancelled by similar factors in the propagators. The same comment does not apply to the spacetime coordinates so that the formula is perfectly compatible with analyticity in the internal bosonic coordinates and anomalous spacetime dimensions.

The analysis of the restrictions imposed by demanding analyticity was carried out in detail for the case of a four-point function in $N = 2$ with four operators of charge two in [45] (this case arises in the $N = 2$ decomposition of the four-point function of $N = 4$ supercurrents, for example). This type of analysis can be expected to be quite complicated for a general correlator. However, it was found that the OPE can help significantly in the analysis of the analyticity properties of four-point functions of one-half BPS operators in $N = 4$ and one would expect that this should also be the case for more complicated correlators [42]. Other related studies of four-point correlators of one-half BPS operators using the OPE can be found in the literature [39, 43, 46, 47].

6. Conclusions

In this paper we have shown that the analytic superspace formalism can be used fruitfully in the study of arbitrary gauge-invariant operators in $N = 4$ SYM. It is a convenient formalism to use for the classification of explicit gauge-invariant operators, simplifies the discussion of reducibility at threshold and allows one to solve for correlation functions in terms of propagators, explicit functions of the coordinates involving the $\mathfrak{gl}(2|2)$ representation matrices, and functions of invariants. We have also shown how to obtain all superinvariants in analytic superspace. In order to complete the analysis of all correlation functions in the theory from the point of view of superconformal invariance one needs to take into account the restrictions imposed by analyticity in the internal even coordinates as we have just mentioned.

As has been shown elsewhere, the formalism can be used to give a compelling non-perturbative argument in support of the non-renormalisation of two- and three-point functions of arbitrary protected operators, including one-quarter BPS series C operators, one-eighth BPS series B operators and short series A operators. However, as we have seen, the situation is more subtle for the case of semi-protected operators which can arise in series A or B. These operators have non-renormalised two-point functions even though the three-point function with an extra supercurrent is not $\mathrm{U}(1)_Y$ invariant. This is because this correlator can have $\mathrm{U}(1)_Y$ charge at most equal to one. Generically, however, we expect that three-point functions involving such an operator together with two protected operators will be subject to renormalisation effects.

The analysis of the properties of two- and three-point functions is greatly facilitated by the use of the $\mathcal{E}$ tensor. This makes it easy to keep track of the $\mathrm{U}(1)_Y$ properties of operators and correlators and is also very useful in the construction of nilpotent superconformal invariants.

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A. From analytic to harmonic superspace

Representations of the superconformal group are perhaps more familiar as superfields on Minkowski superspace or harmonic superspace. In this section we show how one can go from an analytic superfield (generally with superindices) to a harmonic superfield.

Let us suppose we have two coset spaces $M_i = H_i \backslash G, i = 1, 2$ such that $H_1 \subset H_2$. Then M_1 is a fibre bundle over M_2 with fibre $Y = H_1 \backslash H_2$. A tensor field f_2 on M_2, transforming under an induced representation of G, is equivalent to an equivariant field $F : G \to V$, by which we mean $F(h_2 u) = \mathcal{R}(h_2)F(u)$, where $u \in G,\ h_2 \in H_2$ and where $\mathcal{R}$ denotes a linear representation of H_2 on the vector space V. If $s_2(x_2)$ denotes a local section $M_2 \mapsto G$, then the tensor field is given locally by $f_2(x_2) = F(s(x_2))$. We can define a related field $f_1(x_1)$ on M_1 as follows: in local coordinates we can write $x_1 = (x_2, y)$, y being the fibre coordinates, and we can write a local section $s_1 : M_1 \to G$ as $s_1(x_1) = s(y)s_2(x_2)$ where $s(y)$ is a section $Y \to H_2$. We can then put $f_1(x_1) = F(s(x_1)) = F(s(y)s_2(x_2)) = \mathcal{R}(s(y))f_2(x_2)$. This construction gives a field on M_1 which transforms as follows under G:

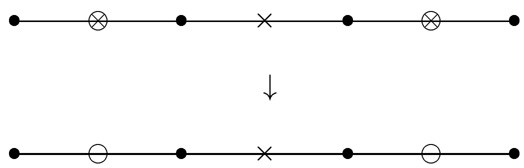
$$f_1(x_1) \mapsto (g \cdot f_1)(x_1) = \mathcal{R}(h_1(x_1, g))f_1(x_1 \cdot g) \tag{A.1}$$

where

$$s(x_1)g = h_1(x_1, g)s(x_1 \cdot g)\,. \tag{A.2}$$

As a tensor under H_1 f_1 will in general transform reducibly, but in the case of interest it is easy to select the required irreducible representation.

We shall now show how to construct fields on $(4, 2, 2)$ harmonic superspace starting from fields on $(4, 2, 2)$ analytic superspace. The relevant fibration is



(A.3)

where the top line represents $(4,2,2)$ harmonic superspace. This diagram indicates that the fibres in this case factorise into left-hand and right-hand parts. These can both be represented by the subdiagram

$$\begin{array}{c} \bullet\!\!-\!\!\!-\!\!\!-\!\!\otimes\!\!-\!\!\!-\!\!\!-\!\!\bullet \\ \downarrow \\ \bullet\!\!-\!\!\!-\!\!\!-\!\!\ominus\!\!-\!\!\!-\!\!\!-\!\!\bullet \end{array} \tag{A.4}$$

The top line here represents the coset space $H\backslash \mathrm{GL}(2|2)$ where H is the set of matrices of the form

$$\left(\begin{array}{cc|cc} \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 \\ \hline \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{array}\right) \tag{A.5}$$

where the bullets represent non-zero elements. The bottom line has no crosses and thus corresponds to a point. So in this context one takes a linear representation of $\mathrm{GL}(2|2)$ and lifts it to the coset. This coset space has only fermionic coordinates so that we are effectively rewriting tensors (or quasi-tensors) as "superfields" which depend on the odd coordinates of the coset but not on the space-time coordinates. We can do this with both factors, take the product and allow the fields so obtained to depend on the coordinates of analytic superspace. In this way we shall have written the latter as fields on harmonic superspace.

A.1 Representations of supergroups as superfields

We shall write an arbitrary finite dimensional irreducible representation of $\mathrm{GL}(2|2)$ as a superfield on the coset space $H\backslash \mathrm{GL}(2|2)$

A coset representative of this space is given by

$$s(\rho)^B{}_A = \left(\begin{array}{c|c} \delta^b{}_a & \rho^\beta{}_a \\ \hline 0 & \delta^\beta{}_\alpha \end{array}\right). \tag{A.6}$$

The Dynkin diagram for this space is

$$\bullet\!\!-\!\!\!-\!\!\!-\!\!\otimes\!\!-\!\!\!-\!\!\!-\!\!\bullet \tag{A.7}$$

and we see that the cross splits the diagram into two $\mathrm{GL}(2)$ representations. We accordingly split the indices A as $A = (\alpha, a)$ with a, α both running from 1 to 2. We shall write a tensor carrying $\mathrm{GL}(2|2)$ indices A as a superfield carrying $\mathrm{GL}(2) \times \mathrm{GL}(2)$ indices (a, α). A representation of $\mathrm{GL}(2|2)$ will decompose into different representations under $\mathrm{GL}(2) \times \mathrm{GL}(2)$ but we need to select the irreducible one. From the triangular structure of the isotropy group one sees that this representation will be the one with the maximum number of a-type indices. A simple example is given by the defining representation V_A. The required field is

$$v_a := V_B (s^{-1})^B{}_a = V_a - V_\beta \rho^\beta_a \tag{A.8}$$

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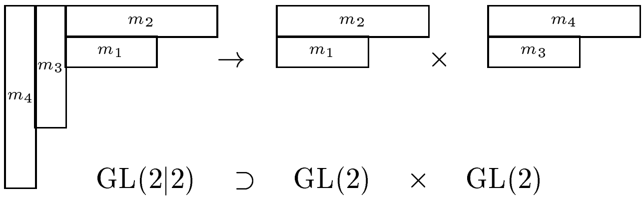
which clearly contains both V_a and V_α. On the other hand $V_B(s^{-1})^B{}_\alpha = V_\alpha$ and is in fact not irreducible under H. (We use the inverse of S in order to get a left representation.) In the general case the index structure of the required superfield can be read off from the Young tableau (A.9):



$$\mathrm{GL}(2|2) \quad \supset \quad \mathrm{GL}(2) \quad \times \quad \mathrm{GL}(2) \tag{A.9}$$

so the superfield has $m_1 + m_2$ α indices and $m_3 + m_4$ a indices arranged according to these Young tableau. One converts from a tensor to a superfield by multiplying the tensor by $m_3 + m_4$ copies of $s^A{}_b$ and $m_1 + m_2$ copies of $s^A{}_\beta$, contracting the tensor indices with the corresponding upstairs indices of $s^A{}_B$, and putting the remaining indices into the correct representation.

Note that the superfield can be straightforwardly converted into a superfield with $m_2 - m_1$ α indices and $m_4 - m_3$ a indices by applying ϵ-tensors. This new superfield will now transform with a $\mathbb{C}$-charge.

As another example consider the symmetric representation V_{AB} with Young tableau $\langle 0, 0, 1, 1\rangle$. It can be written as a "superfield" v_{ab} where

$$v_{ab} = V_{AB}(s^{-1})^B{}_b(s^{-1})^A{}_a(-1)^{(A+a)b} = V_{ab} - 2V_{\beta[a}\rho^\beta{}_{b]} + V_{\alpha\beta}\rho^\alpha_a\rho^\beta_b\,. \tag{A.10}$$

The minus sign can be viewed as follows: we want to contract $(s^{-1})^A{}_a$ directly next to the A index 'before' the B index. Since we can not do this we commute s past the b picking up the minus sign in the process. In this example the a, b indices are anti-symmetric and so one can multiply by ϵ to obtain $v := \epsilon^{ab}v_{ab}$.

Notice that in this example, as for the fundamental representation, the superfields do not have a maximum possible theta expansion, and are thus 'short'. This shortness can be expressed as a constraint on the superfield, which in the above two examples reads

$$D_{\alpha(a}v_{b)} = 0 \qquad D^a_\alpha D^{\alpha b}v = 0 \tag{A.11}$$

where $D^a_\alpha := \partial/\partial\rho^\alpha_a$, and where indices of both types are raised or lowered with epsilon tenors. These are the origins of the quarter BPS constraints and the constraints on short scalar fields which saturate the series A unitary bounds.

In [42] the representations of GL(2|2) were classified into long and short representations, the long ones corresponding to $m_3 > 1$ and the short ones corresponding to $m_3 = 1$ or $m_3 = 0$ (with the canonical choice $m_1 = 0$). On writing the tensor representation as a superfield one finds that the long or typical representations have full superfield expansions (in ρ) whereas the short or atypical representations (such as the ones given in this example) have some terms missing in the theta expansion. For example one can easily convince oneself that the representation with Young tableau $\langle 0, 0, 2, 2\rangle$ has a full ρ expansion and hence is a long representation.

A.2 Analytic superfields as harmonic superfields

We can now apply this construction in order to write superfields on $(4,2,2)$ analytic superspace as superfields on $(4,2,2)$ harmonic superspace. We can simply apply the results of the previous subsection to the left- and right-handed GL$(2|2)$ isotropy groups.

We have the coset representatives

$$s(\rho)^A{}_B = \left(\begin{array}{c|c} \delta^\alpha{}_\beta & \rho^\alpha{}_b \\ \hline 0 & \delta^a{}_b \end{array}\right) \qquad s'(\eta)_{A'}{}^{B'} = \left(\begin{array}{c|c} \delta_{\alpha'}{}^{\beta'} & 0 \\ \hline \eta_{\alpha'}{}^{b'} & \delta_{a'}{}^{b'} \end{array}\right). \tag{A.12}$$

ρ and η will become the extra coordinates that harmonic superspace has in addition to those of analytic superspace. To obtain a superfield on harmonic superspace from one on analytic superspace, simply multiply the analytic superfield on the right by $s^{-1}(\rho)$ and on the left by $s'(\eta)$ in the way described in the previous section and then choose the component with the maximum of internal indices.

For example, a one-half BPS state A has no indices, so it lifts trivially to harmonic superspace. It does not depend on the extra (ρ,η) coordinates as we would expect. The independence of A of (ρ,η) can be expressed in terms of differential constraints:

$$D^a_\alpha A = \bar{D}^{a'}_{\alpha'} A = 0\,, \tag{A.13}$$

where, in the coordinates $(x,\lambda,\pi,y,\rho,\eta)$, we have defined $D^a_\alpha := \partial/\partial\rho^\alpha_a$; $\bar{D}^{a'}_{\alpha'} := \partial/\partial\eta^{\alpha'}_{a'}$.

Now consider a one-quarter BPS operator with Dynkin labels of the form $[001d100]$ where $Q = d+2$ is the Q charge. For any value of d an operator of this type is represented on analytic superspace by a covector field, $V_{A'A}(x,\lambda,\pi,y)$, say. It lifts to a superfield $v_{a'a}(x,\lambda,\pi,y,\rho,\eta)$ in harmonic superspace where

$$\begin{aligned} v_{a'a} &= s'_{a'}{}^{B'} V_{B'B}(s^{-1})^B{}_a \\ &= V_{a'a} - V_{a'\beta}\rho^\beta{}_a + \eta_{a'}{}^{\beta'} V_{\beta' a} - \eta_{a'}{}^{\beta'} V_{\beta'\beta}\rho^\beta{}_a\,. \end{aligned} \tag{A.14}$$

This is also a constrained superfield on harmonic superspace as can be seen by the fact that it has a short ρ expansion. Equation (A.11) shows that the constraints are given by

$$D_{\alpha(a} v_{b)b'} = \bar{D}_{\alpha'(a'} v_{bb')} = 0\,. \tag{A.15}$$

This example makes it clear that we can write any analytic superfield as a harmonic superfield in terms of the coordinates $(x^{\alpha\alpha'},\lambda^{\alpha a'},\pi^{a\alpha'},y^{aa'},\rho^\alpha_a,\eta^{a'\alpha'})$. Whilst analytic superfields are always unconstrained we see that on lifting to harmonic superspace the resulting harmonic superfields often satisfy constraints. The constraints satisfied by irreducible superconformal representations on (a slightly different) harmonic superspace are given in [12].

Harmonic superspace is Minkowski superspace extended by the internal coordinates y and therefore has the standard coordinates of complex super Minkowski space $(x^{\alpha\alpha'}_M,\theta^{\alpha i}$, $\varphi^{\alpha'}_i)$ together with the internal coordinates $y^{aa'}$. If we split the odd Minkowski coordinates in two pairs, $\theta^{\alpha i} = (\theta^{\alpha}_{a}, \theta^{\alpha a'})$ and $\varphi_i^{\alpha'} = (\varphi^{\alpha' a} \varphi^{\alpha'}_{a'})$, then the two sets of coordinates for harmonic superspace are related as follows:

$$\begin{aligned}
x^{\alpha\alpha'} &= x_M^{\alpha\alpha'} + \frac{1}{2}\left(\theta^{\alpha_a}\varphi^{\alpha' a} - \theta^{\alpha a'}\varphi^{\alpha'}_{a'} + 2y^{aa'}\theta^{\alpha}_{a}\varphi^{\alpha'}_{a'}\right)\\
\lambda^{\alpha a'} &= \theta^{\alpha a'} - \theta^{\alpha}_{a} y_{aa'}\\
\pi^{a\alpha'} &= \varphi^{\alpha' a} + y^{aa'}\varphi^{\alpha'}_{a'}\\
\rho^{\alpha}_{a} &= \theta^{\alpha}_{a}\\
\eta^{\alpha'}_{a} &= \varphi^{\alpha'}_{a}\,.
\end{aligned} \tag{A.16}$$

We can rewrite this in the language of GIKOS by introducing a matrix $u_I{}^i \in \mathrm{SL}(4)$. For $(4,2,2)$ harmonic superspace we split the internal coset index into two doublets $I = (r, r')$ and convert SL(4) indices to coset indices using u and its inverse in the standard way. In this notation G-analytic fields depend only on $x^{\alpha\alpha'}, \theta^{\alpha r'}$ and $\varphi^{\alpha'}_{r}$, where

$$x^{\alpha\alpha'} = x_M^{\alpha\alpha'} + \frac{1}{2}(\theta^{\alpha r}\varphi^{\alpha'}_{r} - \theta^{\alpha r'}\varphi^{\alpha'}_{r'})\,. \tag{A.17}$$

The link between the two notations is made by choosing a gauge for u of the form

$$u_I{}^i = \begin{pmatrix} \delta^a{}_b & y^{ab'} \\ 0 & \delta_{a'}{}^{b'} \end{pmatrix} \tag{A.18}$$

Finally, to go to real superspace in the conventions of [26, 27] one needs to replace each Minkowski superspace coordinate z by $-iz$ and then impose the reality conditions, $x = \bar{x}, \varphi = \bar{\theta}$. In the real case the internal symmetry group becomes SU(4). Note also that the x coordinate we are using is the usual one for analytic space which is often denoted x_A in the literature.

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